CLOSED SURFACES AND CHARACTER VARIETIES

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ABSTRACT. The powerful character variety techniques of Culler and Shalen can be used to find essential surfaces in knot manifolds. We show that module structures on the coordinate ring of the character variety can be used to identify detected boundary slopes as well as when closed surfaces are detected. This approach also yields new number theoretic invariants for the character varieties of knot manifolds.

1. INTRODUCTION

Suppose that N is a compact, irreducible 3-manifold with torus boundary and that X is an irreducible algebraic component of the $\mathrm{SL}_2\mathbb{C}$-character variety for N. It is known that the dimension of X is at least one [6]. It is also known that X contains a great deal of topological information about N.

Let $\widetilde{\mathsf{X}} \to \mathsf{X}$ be a birational map from a smooth projective curve. This map is defined for all but finitely many points. These points are called ideal points. For $\gamma \in \pi_1 N$ and $\chi \in \mathsf{X}$, define $I_\gamma(\chi) = \chi(\gamma)$. This determines a rational function $I_\gamma \colon \widetilde{\mathsf{X}} \to \mathbb{C}$. This paper concerns the following landmark result of Culler and Shalen.

Theorem 1 (Culler-Shalen). *If $\hat{x}$ is an ideal point, there is an associated non-empty essential surface Σ in N.*

(1) *Σ may be chosen to have empty boundary if and only if I_α is regular at $\hat{x}$ for every peripheral element $\alpha \in \pi_1 N$.*

(2) *Otherwise, there exists a unique slope with the property that if $\alpha \in \pi_1 N$ represents this slope then I_α is regular at $\hat{x}$. In this case, every component of $\partial\Sigma$ represents the slope corresponding to α.*

When an essential surface arises from X and this theorem, we say that the surface is detected by X.

Since 1983, case (2) of Theorem 1 has been carefully studied. Many famous papers have provided both applications and insight for this case. Notably, the Culler-Shalen norm of [9] and the A-polynomial of [6] can tell us exactly which slopes are detected. These tools are remarkably effective and have been used in proofs of several famous theorems including the Smith Conjecture [22] and the Cyclic Surgery Theorem [9]. In contrast, there is very little in the literature concerning case (1) of Theorem 1.

It has also been difficult to interpret global properties of X in terms of the topology of N. This paper introduces new functions which relate detected boundary slopes and essential closed surfaces in the manifold to module structures of the coordinate ring $\mathbb{C}[\mathsf{X}]$. In the spirit of [7], these functions also contain number theoretic information about X.

The trace ring $T(\mathsf{X})$ is the subring of the coordinate ring $\mathbb{C}[\mathsf{X}]$ which is generated by $\{I_\gamma \,|\, \gamma \in \pi_1 N\}$. Define $T_\mathbb{Q}(\mathsf{X})$ to be the smallest $\mathbb{Q}$-algebra in $\mathbb{C}[\mathsf{X}]$ which contains $T(\mathsf{X})$. Suppose that $\alpha \in \pi_1 N$ is primitive and peripheral. Then $\mathbb{C}[\mathsf{X}]$, $T_\mathbb{Q}(\mathsf{X})$, and $T(\mathsf{X})$ have the structure of a $\mathbb{C}[I_\alpha]$-module, a $\mathbb{Q}[I_\alpha]$-module, and a $\mathbb{Z}[I_\alpha]$-module, respectively. Let $\mathrm{Rk}_\mathsf{X}^\mathbb{C}(\alpha)$, $\mathrm{Rk}_\mathsf{X}^\mathbb{Q}(\alpha)$, and $\mathrm{Rk}_\mathsf{X}^\mathbb{Z}(\alpha)$ denote the ranks of these modules. Note that the functions I_α are well-defined on slopes. Hence, these rank functions may be viewed as functions on $\mathcal{S}$, the set of slopes for N. It is clear that

$$(1) \qquad\qquad \mathrm{Rk}_\mathsf{X}^\mathbb{C} \;\leq\; \mathrm{Rk}_\mathsf{X}^\mathbb{Q} \;\leq\; \mathrm{Rk}_\mathsf{X}^\mathbb{Z}.$$

The following theorem is the main theorem of this paper.

Theorem 2. *Let $\mathbb{X} \in \{\mathbb{C}, \mathbb{Q}, \mathbb{Z}\}$. Then*
   (1) *The function $\mathrm{Rk}_\mathsf{X}^\mathbb{X} \colon \mathcal{S} \to \mathbb{Z}^+ \cup \{\infty\}$ is constant with value ∞ if and only if X detects a closed essential surface.*
   (2) *Otherwise, $\mathrm{Rk}_\mathsf{X}^\mathbb{X}(\alpha) = \infty$ if and only if X detects the slope α.*

We begin by proving the theorem for $\mathbb{X} = \mathbb{C}$. A straightforward application of this case yields a proposition of independent interest.

Proposition 1.1. *Let X° be the union of the irreducible components X' of the $SL_2\mathbb{C}$ character variety such that*
   (1) *X' detects a closed essential surface if and only if X does, and*
   (2) *for each slope α, X' detects a surface with boundary slope α if and only if X does.*
Then X° is defined over $\mathbb{Q}$.

This proposition, together with Theorem 1, also allows us to improve the result to include $\mathbb{X} = \mathbb{Q}$. As an application of this result, we prove another proposition.

Proposition 1.2. *If X contains the character of a non-integral representation which detects a closed essential surface, then X detects a closed essential surface.*

Together with data compiled by Goodman, Heard, and Hodgson, this proposition gives many examples of hyperbolic knot manifolds for which the hyperbolic components of their character varieties detect closed essential surfaces. We also use this proposition in our proof that the main theorem holds in the case $\mathbb{X} = \mathbb{Z}$.

In addition to our main result, we begin an investigation of the basic properties of the functions $\mathrm{Rk}_\mathsf{X}^\mathbb{X}$ and include computations for several examples. It is easy to prove the following two propositions.

Proposition 1.3. *Suppose that N is a knot manifold and $H_1(N; \mathbb{Z}) \cong \mathbb{Z}$, generated by α. Let $\mathsf{X}_A \subseteq \mathsf{X}(N)$ be the curve consisting of all abelian representations of $\pi_1 N$. Then*

$$\mathrm{Rk}_{\mathsf{X}_A}^\mathbb{C}(\alpha) \;=\; \mathrm{Rk}_{\mathsf{X}_A}^\mathbb{Z}(\alpha) \;=\; 1.$$

Proposition 1.4. *If N is a knot manifold and X is a norm curve component of $\mathsf{X}(N)$ then $\mathrm{Rk}_\mathsf{X}^\mathbb{C}(\alpha) \geq 2$ for every slope α.*

As a corollary to Theorem 2, we have the following.

Corollary 1.5. *Suppose N is the exterior of a two-bridge knot and $\langle \mu, \beta \,|\, \omega\beta = \mu\omega \rangle$ is the standard presentation for $\pi_1 N$. If $\mathsf{X} \subset \mathsf{X}(N)$ is an irreducible algebraic*

component defined over $\mathbb{Q}$ then X is defined by an irreducible polynomial of the form

$$I_{\mu\beta}^n - \sum_{j=0}^{n-1} p_j(I_\mu)I_{\mu\beta}^j.$$

The set $\{I_{\mu\beta}^j\}_0^{n-1}$ is a free basis for $\mathbb{C}[X]$ as a $\mathbb{C}[I_\mu]$-module, $T_\mathbb{Q}(\mathsf{X})$ as a $\mathbb{Q}[I_\mu]$-module, and $T(\mathsf{X})$ as a $\mathbb{Z}[I_\mu]$-module.

We point out that, with the appropriate definitions, these results also hold for the $\mathrm{PSL}_2\mathbb{C}$-character variety. In what follows, we use X to indicate that we are working in the $\mathrm{SL}_2\mathbb{C}$ setting and Y in the $\mathrm{PSL}_2\mathbb{C}$ setting. When $N \subset S^3$ is a knot exterior, we reserve $\lambda, \mu \in \pi_1 N$ to be a longitude, meridian pair. We write X_A and Y_A for the algebraic sets of abelian characters. When N is hyperbolic, we use the notation X_0 and Y_0 to indicate algebraic components of the character varieties which contain a discrete faithful character.

Below, we list the results of our calculations in five different examples. The last example is of particular interest. It shows that the inequalities (1) need not be equalities and that, although $T(\mathsf{X})$ must always be torsion free as a $\mathbb{Z}[I_\sigma]$-module, it need not be a free module.

(1) Suppose N is the exterior of a trefoil knot. Then $\mathsf{X}(N) = \mathsf{X}_A \cup \mathsf{X}_0$ where X_0 is irreducible. We have

$$\mathrm{Rk}_{\mathsf{X}_0}^\mathbb{Z}(\mu) = 1.$$

This shows that the converse to Proposition 1.3 does not hold.

(2) Suppose N is the exterior of the figure-eight knot. We have

$$\mathrm{Rk}_{\mathsf{X}_0}^\mathbb{Z}(\mu) = \mathrm{Rk}_{\mathsf{Y}_0}^\mathbb{Z}(\mu) = 2$$
$$\mathrm{Rk}_{\mathsf{Y}_0}^\mathbb{C}(\lambda) = \mathrm{Rk}_{\mathsf{Y}_0}^\mathbb{Z}(\lambda) = 4$$
$$\mathrm{Rk}_{\mathsf{Y}_0}^\mathbb{C}(\mu^2\lambda) = \mathrm{Rk}_{\mathsf{Y}_0}^\mathbb{Z}(\mu^2\lambda) = 5.$$

(3) Suppose $N = M_{003}$. Then $\pi_1 N = \langle \gamma, \eta \mid \gamma\eta\gamma^{-2}\eta\gamma\eta^3 \rangle$. Define $\mu = (\eta^2\gamma\eta\gamma)^{-1}$ and $\lambda = (\gamma\eta\gamma)^{-1}\eta\gamma\eta$. Then μ and λ are primitive, peripheral, and generate a peripheral subgroup of N. We have

$$\mathrm{Rk}_{\mathsf{X}_0}^\mathbb{C}(\mu) = \mathrm{Rk}_{\mathsf{X}_0}^\mathbb{Z}(\mu) = 4.$$

Also, $I_\mu \in T(\mathsf{Y}_0)$ and

$$\mathrm{Rk}_{\mathsf{Y}_0}^\mathbb{C}(\mu) = \mathrm{Rk}_{\mathsf{Y}_0}^\mathbb{Z}(\mu) = 2.$$

Note that, since Y_0 is a norm curve, these ranks achieve their minimum possible value. In contrast to examples (1) and (2), the $\mathrm{PSL}_2\mathbb{C}$-rank is strictly smaller than the $\mathrm{SL}_2\mathbb{C}$-rank (at μ). As with examples (1) and (2), all $\mathbb{Z}[I_\mu]$-modules are free.

(4) Suppose N is the exterior of the knot 8_{20}. N is hyperbolic, so we consider X_0. The diagram in Figure 1 gives a Wirtinger presentation for $\pi_1 N$ which reduces to $\langle \mu, \gamma \mid \mu\gamma(\mu\gamma\mu)^{-1}\gamma^3(\mu\gamma\mu)^{-1}\gamma\mu\gamma^{-2} \rangle$ where $\gamma = \eta\sigma$.

The functions I_μ, I_γ, and $I_{\gamma\mu}$ give an embedding of X_0 into $\mathbb{C}^3$. I_γ and $I_{\gamma\mu}$ both satisfy irreducible integral dependencies in $\mathbb{Z}[I_\mu][x]$ of degree 5. It follows that $\mathcal{B} = \{I_\gamma^i I_{\gamma\mu}^j \mid 0 \le i, j \le 4\}$ is a generating set for $\mathbb{C}[X_0]$ as
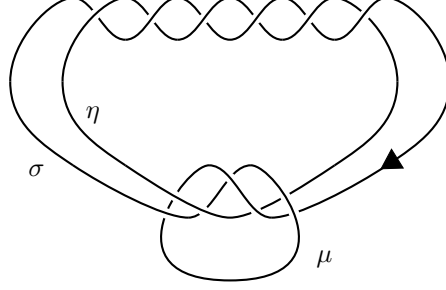
FIGURE 1. The knot 8_{20} labelled with Wirtinger generators

a $\mathbb{C}[I_\mu]$-module, $T_\mathbb{Q}(\mathsf{X}_0)$ as a $\mathbb{Q}[I_\mu]$-module, and $T(\mathsf{X}_0)$ as a $\mathbb{Z}[I_\mu]$-module. Notice that $|\mathcal{B}| = 25$.

A Groebner basis argument shows that relations amongst the elements of $\mathcal{B}$ are plentiful and, in fact, $\{1, I_\gamma, I_\gamma^2, I_\gamma^3, I_\gamma^2 I_{\gamma\mu}, I_{\gamma\mu}\}$ is a free basis for each of the above modules. Hence,

$$\mathrm{Rk}_{\mathsf{X}_0}^\mathbb{C}(\mu) \;=\; \mathrm{Rk}_{\mathsf{X}_0}^\mathbb{Z}(\mu) \;=\; 6$$

and $T(\mathsf{X}_0)$ is a free $\mathbb{Z}[I_\mu]$-module.

(5) Suppose N is the once punctured hyperbolic torus bundle from Section 5 of [11]. Then $\pi_1 N = \langle \alpha, \beta, \tau \,|\, \tau\alpha\tau^{-1} = (\beta\alpha\beta)^{-1}, \tau\beta\tau^{-1} = \beta\alpha(\beta\alpha\beta)^{-3} \rangle$. The elements τ and $\lambda = [\alpha, \beta]$ form a basis for the peripheral subgroup of N and the functions

$$\begin{array}{llll} t = I_\tau & u = I_{\alpha\tau} & v = I_{\beta\tau} & w = I_{\alpha\beta\tau} \\ x = I_\alpha & y = I_\beta & z = I_{\alpha\beta} & \end{array}$$

give an embedding of $\mathsf{X}(N)$ into $\mathbb{C}^7$. For $\epsilon \in \{0, 1\}$, we have an irreducible algebraic component $\mathsf{X}_\epsilon \subset \mathsf{X}(N)$ which contains a discrete faithful character.

  (a) $\{1, z, z^2, z^3, y, zy, z^2 y, z^3 y\}$ is a free basis for $\mathbb{C}[\mathsf{X}_\epsilon]$ as a $\mathbb{C}[I_\lambda]$-module. Hence, $\mathrm{Rk}_{\mathsf{X}_\epsilon}^\mathbb{C}(\lambda) = 8$.
  (b) $\{1, z, z^2, z^3, y, zy, z^2 y, z^3 y, t, zt, z^2 t, z^3 t, yt, zyt, z^2 yt, z^3 yt\}$ is a free basis for $T(\mathsf{X}_\epsilon)$ as a $\mathbb{Q}[I_\lambda]$-module. Hence, $\mathrm{Rk}_{\mathsf{X}_\epsilon}^\mathbb{Q}(\lambda) = 16$.
  (c) $\{1, y, y^2, y^3, t, u, v, w, x, z, yt, yu, yx, y^2 x, xt, xu, vy\}$ generates $T(\mathsf{X}_\epsilon)$ as a $\mathbb{Z}[I_\lambda]$-module. However, $T(\mathsf{X}_\epsilon)$ is torsion free but $\underline{\text{not}}$ free as a $\mathbb{Z}[I_\lambda]$-module. In particular, $\mathrm{Rk}_{\mathsf{X}_\epsilon}^\mathbb{Z}(\lambda) = 17$.

The projection $(t, y, z)\colon \mathsf{X}_0 \cup \mathsf{X}_1 \to \mathbb{C}^3$ is an isomorphism onto its image and the relation

$$\mathrm{Rk}_{\mathsf{X}_\epsilon}^\mathbb{Q}(\lambda) < \mathrm{Rk}_{\mathsf{X}_\epsilon}^\mathbb{Z}(\lambda)$$

reflects the fact that the inverse of this isomorphism is not defined over $\mathbb{Z}$. The equality

$$\mathrm{Rk}_{\mathsf{X}_\epsilon}^\mathbb{Q}(\lambda) = 2 \cdot \mathrm{Rk}_{\mathsf{X}_\epsilon}^\mathbb{C}(\lambda)$$

reflects the fact that X_ϵ is not defined over $\mathbb{Q}$.

2. Algebraic Geometry

Throughout this section we take k to be an algebraically closed field. All varieties are defined over k. When X is an irreducible variety we write $k[X]$ and $k(X)$ to denote the ring of regular functions on X and the function field for X, respectively. If $p \in X$, define $\mathcal{O}_{X,p}$ to be the ring of germs of functions which are regular on neighborhoods of p. We have a surjective homomorphism $k[X] \to k$ given by $f \mapsto f(p)$. Let $\mathfrak{m}_p$ be the maximal ideal which is the kernel of this map. By Theorem 3.2 part (c) of [13], $\mathcal{O}_{X,p}$ is isomorphic to the localization $k[X]_{\mathfrak{m}_p}$. We use this isomorphism to identify these two rings and think of $\mathcal{O}_{X,p}$ as a subring of $k(X)$.

Definitions 2.1. Suppose $A \subseteq B$ are commutative rings and $1_B \in A$.

(1) An element $b \in B$ is *integral over* A if there is a number $n \in \mathbb{Z}^+$ and $\{\alpha_i\}_0^{n-1} \subset A$ such that

$$b^n + \alpha_{n-1}b^{n-1} + \cdots + \alpha_0 \ = \ 0.$$

This equation is called an *integral dependence relation of b over A*.

(2) B *is integral over* A if every element of B is integral over A.

(3) The *integral closure* $\overline{A}$ *of A in B* is the set of all elements of B which are integral over A.

(4) If $\overline{A} = A$ then A is *integrally closed in B*.

By Corollary 5.3 of [1], the integral closure of A in B is a ring. The following characterization of integral closures is useful.

Theorem 3 (Corollary 5.22 of [1]). *Suppose A is a subring of a field K. The integral closure of A in K is the intersection of all valuation rings of K that contain A.*

Definitions 2.2. Suppose that X is an affine variety.

(1) X is a *normal variety* if $k[X]$ is integrally closed in $k(X)$.

(2) A *normalization of* X is an irreducible normal variety X^ξ and a regular birational map $\xi \colon X^\xi \to X$ where $k[X^\xi]$ is integral over $\xi^*(k[X])$.

If X is an affine variety and $p \in X$ then the quotient $\mathfrak{m}_p/\mathfrak{m}_p^2$ has the structure of a k-vector space.

Definitions 2.3. Suppose that X is an affine variety.

(1) A point $p \in X$ is *non-singular* if $\dim\left(\mathfrak{m}_p/\mathfrak{m}_p^2\right)$ equals the dimension of X.

(2) X is *non-singular* if every point in X is non-singular.

Theorem 1 in Chapter II, Section 5.1 of [21] states that non-singular affine varieties are normal.

It is clear that if $\mathfrak{m}_p$ is principal then $\dim\left(\mathfrak{m}_p/\mathfrak{m}_p^2\right) = 1$. Theorem 2 in Chapter II, Section 5.1 of [21] implies that if X is a normal affine algebraic curve and $p \in X$ then $\mathfrak{m}_p$ is principal. Thus, we have the following well-known theorem.

Theorem 4. *Suppose X is an irreducible affine algebraic curve. X is non-singular if and only if X is normal.*

The following theorem is a direct consequence of Theorems 4 and 5 in Chapter II, Section 5.2 in [21] (and the first part of the proof of Theorem 4).

Theorem 5. *If* X *is an irreducible affine algebraic curve then* X *has a normalization* $\xi\colon \mathsf{X}^\xi \to \mathsf{X}$*. Moreover, the normalization is unique, affine, and its coordinate ring is the integral closure of* $\xi^*\left(k[\mathsf{X}]\right)$*.*

The projective coordinates on projective space $\mathbb{P}^n$ give distinguished open affine subsets $\{U_i\}_0^n$ which cover $\mathbb{P}^n$. So if $\mathsf{X} \subset \mathbb{P}^n$ is an irreducible projective variety then we have the distinguished affine open subsets $U_i \cap \mathsf{X}$ which cover X. We say that X is non-singular if $U_i \cap \mathsf{X}$ is non-singular for every $i = 0, \ldots, n$.

Definition 2.4. Suppose that X is an irreducible affine curve and let $\xi\colon \mathsf{X}^\xi \to \mathsf{X}$ the normalization of X. As in [13, Chapter 6], there is a smooth projective curve $\widetilde{\mathsf{X}}$ (unique up to isomorphism) so that X^ξ is isomorphic to an open set in $\widetilde{\mathsf{X}}$. The projective variety $\widetilde{\mathsf{X}}$ is called the *smooth projective model for* X.

Henceforth, we identify X^ξ with its image in $\widetilde{\mathsf{X}}$ and we let $\iota\colon \widetilde{\mathsf{X}} \dashrightarrow \mathsf{X}^\xi$ be the rational map which is the identity on X^ξ.

Definition 2.5. The *ideal points* of X are the points in the set $\mathcal{I}(\mathsf{X}) = \widetilde{\mathsf{X}} - \mathsf{X}^\xi$. It follows from the uniqueness of normalizations and smooth projective models that $\mathcal{I}(\mathsf{X})$ is well-defined.

We have dominant birational maps

$$\widetilde{\mathsf{X}} \overset{\iota}{\dashrightarrow} \mathsf{X}^\xi \overset{\xi}{\longrightarrow} \mathsf{X}$$

The induced maps

$$k\big(\widetilde{\mathsf{X}}\big) \overset{\iota^*}{\longleftarrow} k\left(\mathsf{X}^\xi\right) \overset{\xi^*}{\longleftarrow} k(\mathsf{X})$$

are isomorphims. Moreover, $\xi^*\big(k[\mathsf{X}]\big) \subseteq k[\mathsf{X}^\xi]$.

Now suppose that Y is an affine variey and $\varphi\colon \mathsf{X} \to Y$ is a regular map with $\overline{\varphi(\mathsf{X})} = Y$. Then $\varphi^*\colon k(Y) \to k(\mathsf{X})$ is a field monomorphism and $\varphi^*(k[Y]) \subseteq k[\mathsf{X}]$.

Definition 2.6. A *hole* in $\varphi\colon \mathsf{X} \to Y$ is a point $\hat{p} \in \mathcal{I}(\mathsf{X})$ such that $(\varphi\xi\iota)^*\left(k[Y]\right) \subseteq \mathcal{O}_{\widetilde{\mathsf{X}},\hat{p}}$.

Remark 1. *If* $\hat{p}$ *is a hole in* $\varphi\colon \mathsf{X} \to Y$ *then, since* $\hat{p} \notin \mathsf{X}^\xi$*, there is an element* $f \in k[\mathsf{X}]$ *with* $(\xi\iota)^*(f) \notin \mathcal{O}_{\widetilde{\mathsf{X}},\hat{p}}$ *. Also, if* φ *is a birational map, then* φ *is surjective if and only if* φ *has no hole.*

Lemma 2.7. *If* $\varphi\colon \mathsf{X} \to Y$ *has a hole then* $k[\mathsf{X}]$ *is not integral over* $\varphi^*\left(k[Y]\right)$*.*

Proof. Take a distinguished open affine set $U_i \subset \mathbb{P}^n$ with $\hat{p} \in U_i$. Set $\mathsf{X}_{\hat{p}} = U_i \cap \widetilde{\mathsf{X}}$. By Theorem 4, $\mathsf{X}_{\hat{p}}$ is a non-singular affine curve so $k[\mathsf{X}_{\hat{p}}]$ is integrally closed. Recall that $\mathcal{O}_{\mathsf{X}_{\hat{p}},\hat{p}}$ is isomorphic to the localization $k[\mathsf{X}_{\hat{p}}]_{\mathfrak{m}_{\hat{p}}}$. Proposition 5.13 of [1] gives that this localization is integrally closed. The inclusion induced isomorphism $k(\widetilde{\mathsf{X}}) \to k(\mathsf{X}_{\hat{p}})$ restricts to an isomorphism $\mathcal{O}_{\widetilde{\mathsf{X}},\hat{p}} \to \mathcal{O}_{\mathsf{X}_{\hat{p}},\hat{p}}$. Hence $\mathcal{O}_{\widetilde{\mathsf{X}},\hat{p}}$ is integrally closed.

The point $\hat{p}$ is a hole in $\varphi\colon \mathsf{X} \to Y$ so $(\varphi\xi\iota)^*(k[Y]) \subseteq \mathcal{O}_{\widetilde{\mathsf{X}},\hat{p}}$. Since $\mathcal{O}_{\widetilde{\mathsf{X}},\hat{p}}$ is integrally closed, every element of $k(\widetilde{\mathsf{X}})$ which is integral over $(\varphi\xi\iota)^*(k[Y])$ is an element of $\mathcal{O}_{\widetilde{\mathsf{X}},\hat{p}}$. We have $f \in k[\mathsf{X}]$ with $(\xi\iota)^*(f) \notin \mathcal{O}_{\widetilde{\mathsf{X}},\hat{p}}$, hence $(\xi\iota)^*(k[\mathsf{X}])$ is not integral over $(\varphi\xi\iota)^*(k[Y])$ and so $k[\mathsf{X}]$ is not integral over $\varphi^*(k[Y])$. $\square$

Theorem 6. *Suppose* X *is an irreducible affine algebraic curve and* $\varphi\colon \mathsf{X} \to Y$ *is a regular map with* $\overline{\varphi(\mathsf{X})} = Y$. *Then* φ *has no hole if and only if* $k[\mathsf{X}]$ *is integral over* $\varphi^*(k[Y])$.

Proof. By Lemma 2.7, we need only show that if φ has no hole then $k[\mathsf{X}]$ is integral over $\varphi^*(k[Y])$.

Assume, to the contrary, that $k[\mathsf{X}]$ is not integral over $\varphi^*(k[Y])$. Since ξ^* is injective, this implies that $k[\mathsf{X}^\xi]$ is not integral over $(\varphi\xi)^*(k[Y])$. Taking integral closures in $k(\mathsf{X}^\xi)$ we have

$$k[\mathsf{X}^\xi] \;=\; \overline{k[\mathsf{X}^\xi]} \;\supsetneqq\; \overline{(\varphi\xi)^*(k[Y])}.$$

Take $f \in k[\mathsf{X}^\xi] - \overline{(\varphi\xi)^*(k[Y])}$. By Theorem 3, there is a valuation ring $R \subset k(\mathsf{X}^\xi)$ with $f \notin R$ and $k \subset \overline{(\varphi\xi)^*(k[Y])} \subseteq R$.

<u>Claim:</u> $\iota^*(R) = \mathcal{O}_{\widetilde{\mathsf{X}},\hat{p}}$ for some $\hat{p} \in \widetilde{\mathsf{X}}$.

Using the claim, $\iota^*(f) \notin \mathcal{O}_{\widetilde{\mathsf{X}},\hat{p}}$ so $\hat{p} \in \mathcal{I}(\mathsf{X})$. Moreover, $(\varphi\xi)^* k[Y] \subseteq R$ so $(\varphi\xi\iota)^*(k[Y]) \subseteq \iota^*(R) = \mathcal{O}_{\widetilde{\mathsf{X}},\hat{p}}$. That is, $\hat{p}$ is a hole in φ, a contradiction.

The claim follows from Corollary 6.6 of [13] since it gives an open set X_R in $\widetilde{\mathsf{X}}$ and a point $\hat{p} \in \mathsf{X}_R$ such that $\iota^*(R) = \mathcal{O}_{\mathsf{X}_R,\hat{p}} = \mathcal{O}_{\widetilde{\mathsf{X}},\hat{p}}$. $\qquad\square$

It is well-known that, in this setting, $k[\mathsf{X}]$ is integral over $f^*(k[Y])$ if and only if $k[\mathsf{X}]$ is a finitely generated $f^*(k[Y])$-module (see for example Proposition 5.1 and Corollary 5.2 of [1]). Hence, we have the following immediate corollary.

Corollary 2.8. *Suppose* X *is an irreducible affine algebraic curve and* $\varphi\colon \mathsf{X} \to Y$ *is a regular map with* $\overline{\varphi(\mathsf{X})} = Y$. *Then* φ *has no hole if and only if* $k[\mathsf{X}]$ *is a finitely generated* $\varphi^*(k[Y])$-*module.*

Remarks 1. *Suppose that* $k[\mathsf{X}]$ *is a finitely generated* $\varphi^*(k[Y])$-*module.*

(1) *Since* X *is irreducible,* $k[\mathsf{X}]$ *has no zero divisors. Hence* $k[\mathsf{X}]$ *is a torsion free* $\varphi^*(k[Y])$-*module.*

(2) *We can be more concrete about a basis for* $k[\mathsf{X}]$ *as a* $\varphi^*(k[Y])$-*module. Let* $\{x_i\}_1^m$ *be coordinate functions for* X. *By Theorem 6, each* x_i *is integral over* $\varphi^*(k[Y])$. *Let* n_i *be the degree of an integral dependence for* x_i *and define* $\mathcal{S} = \{x_1^{\alpha_1} \cdots x_m^{\alpha_m} \,|\, 0 \le \alpha_i < n_i\}$. *Every element of* $k[\mathsf{X}]$ *may be expressed as a* $\varphi^*(k[\mathsf{X}])$-*linear combination of the elements from the finite set* $\mathcal{S}$. *(See Proposition 2.16 of* [1]*.)*

3. CHARACTER VARIETIES, BOUNDARY SLOPES, AND CLOSED ESSENTIAL SURFACES

Definition 3.1. A *knot manifold* is a connected, compact, irreducible, orientable 3-manifold whose boundary is an incompressible torus.

Let N be a knot manifold and $\Gamma = \pi_1(N)$. Denote the set of $\mathrm{SL}_2\mathbb{C}$-representations of Γ as $\mathsf{R}(N)$ and the set of characters of representations in $\mathsf{R}(N)$ as $\mathsf{X}(N)$. Let $t\colon \mathsf{R}(N) \to \mathsf{X}(N)$ be the map which takes representations to their characters. In [10], it is shown that $\mathsf{R}(N)$ and $\mathsf{X}(N)$ are affine algebraic sets defined over $\mathbb{C}$ and the map t is regular. We will refer to $\mathsf{R}(N)$ and $\mathsf{X}(N)$ as the *representation variety* and *character variety* for N.

Culler and Shalen have revealed deep connections between essential surfaces in N and the character variety $\mathsf{X}(N)$. We outline some of their results in what follows. For more background, see [10], the survey article [23], or Chapter 1 of [9].

Definitions 3.2. Suppose that X is a non-empty algebraic subset of $\mathsf{X}(N)$.

(1) Given $\gamma \in \Gamma$, the *trace function* for γ on X is the regular function $I_\gamma \in \mathbb{C}[\mathsf{X}]$ defined by $I_\gamma(\chi) = \chi(\gamma)$.
(2) Let $T(\mathsf{X})$ be the subring (with 1) in $\mathbb{C}[\mathsf{X}]$ generated by $\{I_\gamma | \gamma \in \Gamma\}$. $T(\mathsf{X})$ is called the *trace ring* for X.

The following proposition gives that, as a ring, $T(\mathsf{X})$ is finitely generated.

Proposition 3.3 (Proposition 4.4.2 of [23]). *Let $\{\gamma_i\}_1^n$ be a generating set for Γ. Then $T(\mathsf{X})$ is generated, as a ring, by the constant function 1 along with the functions in the set*

$$\left\{ I_V \,\middle|\, V = \gamma_{j_1} \cdots \gamma_{j_k} \text{ where } 1 \le k \le n \text{ and } 1 \le j_1 < \cdots < j_k \le n \right\}.$$

The smallest $\mathbb{C}$-algebra containing $T(\mathsf{X})$ is the coordinate ring $\mathbb{C}[\mathsf{X}]$. Therefore, a generating set $\{\gamma_i\}_1^n$ for Γ gives an embedding of X into $\mathbb{C}^{2^n-1}$ by taking the functions I_V to be coordinate functions. It is straightforward to see that under this embedding, $\mathsf{X}(N)$ is cut out by polynomials with coefficients in $\mathbb{Z}$.

We have a regular map $\partial \colon \mathsf{X}(N) \to \mathsf{X}(\partial N)$ from $\mathsf{X}(N)$ to the character variety for the peripheral subgroup (well-defined up to conjugation) of Γ given by restricting characters. For a non-empty algebraic subset X of $\mathsf{X}(N)$, let $\partial\mathsf{X}$ denote the Zariski closure of $\partial(\mathsf{X})$.

Definitions 3.4.

(1) The unoriented isotopy class of an essential simple closed curve in ∂N is called a *slope*. We denote the set of slopes on N as $\mathcal{S}$.
(2) A *boundary slope* is a slope which is realized as a boundary component of an essential surface in N.

Each slope corresponds to a pair $\{\pm a\} \subset H_1(\partial N; \mathbb{Z})$. The inverse of the Hurewicz isomorphism is an isomorphism $H_1(\partial N; \mathbb{Z}) \to \pi_1 \partial N$. This gives a monomorphism $e \colon H_1(\partial N; \mathbb{Z}) \to \Gamma$ which is well-defined up to conjugation. Since traces are invariant under inverses and conjugation, the function $\{\pm a\} \mapsto I_{e(a)}$ is well-defined. When $\alpha = \{\pm a\}$ is a slope we write $I_\alpha = I_{e(a)}$.

Assume now that $\mathsf{X} \subseteq \mathsf{X}(N)$ is an irreducible affine curve and let

$$\widetilde{\mathsf{X}} \overset{\iota}{\dashrightarrow} \mathsf{X}^\xi \overset{\xi}{\longrightarrow} \mathsf{X}$$

be the corresponding maps and varieties as defined in Section 2. The following theorem is well known and fundamental, see Theorem 2.2.1 and Proposition 2.3.1 of [10]. It is a translation of Theorem 1 into the language of Sections 2 and 3 of this paper.

Theorem 7 (Culler-Shalen). *For every ideal point $\hat{x}$ of X, there is an associated non-empty essential surface Σ in N.*

(1) *Σ may be chosen to have empty boundary if and only if*

$$(\partial \xi \iota)^* \left(\mathbb{C}[\partial \mathsf{X}] \right) \subseteq \mathcal{O}_{\widetilde{\mathsf{X}}, \hat{x}}.$$

(2) *Otherwise, there exists a unique slope α such that $I_\alpha \in \mathcal{O}_{\widetilde{\mathsf{X}},\hat{x}}$. In this case, every component of $\partial\Sigma$ represents the slope α.*

Theorem 7 is broadly applicable. For instance, if N is the exterior of a non-trivial knot in S^3 then Kronheimer and Mrowka show that $\mathsf{X}(N)$ contains a curve with infinitely many irreducible characters [16]. Also, if the interior of N admits a complete hyperbolic structure with finite volume and χ_ρ is the character of a discrete faithful representation then there is a unique algebraic component $\mathsf{X}_0 \subseteq \mathsf{X}(N)$ which contains χ_ρ. Furthermore, X_0 is a curve (see [24] and [23]). We refer to such a curve as a *hyperbolic curve*.

Definitions 3.5.

(1) Suppose that $\mathsf{X} \subseteq \mathsf{X}(N)$ is an irreducible affine curve and $\hat{x}$ is an ideal point of X. A non-empty essential surface $\Sigma \subset N$ is *associated to* $\hat{x}$ if it is contained in a surface given by $\hat{x}$ and Theorem 7.
(2) Suppose that X is an algebraic subset of $\mathsf{X}(N)$. A surface Σ is *detected by* X if there is an ideal point $\hat{x}$ of an irreducible affine curve in X so that Σ is associated to $\hat{x}$.
(3) If an essential surface $\Sigma \subset N$ is detected by X (associated to $\hat{x}$) and $\partial\Sigma \neq \emptyset$ the boundary slope represented by a component of $\partial\Sigma$ is *detected by* X (associated to $\hat{x}$). The slope is *strongly* or *weakly* detected (associated) depending on whether $\hat{x}$ satisfies case (1) or case (2) of Theorem 7, respectively.

Remark 2. *It is natural to ask if, whenever N contains a closed essential surface, there is a closed essential surface in N detected by $\mathsf{X}(N)$. The author is not certain if the answer is known to be no, however there are compelling reasons to believe that the answer is no, see for example [5] or Remark 5.1 in [11].*

We conclude this section with some applications of the work in Section 2.

Theorem 8. *Suppose N is a knot manifold and X is an irreducible algebraic subset of $\mathsf{X}(N)$. The following are equivalent.*

(1) X *does not detect a closed essential surface.*
(2) $\dim(\mathsf{X}) = \dim(\partial\mathsf{X}) = 1$ *and* $\partial\colon \mathsf{X} \to \partial\mathsf{X}$ *does not have a hole.*
(3) $\mathbb{C}[\mathsf{X}]$ *is integral over* $\partial^*(\mathbb{C}[\partial\mathsf{X}])$.
(4) $\mathbb{C}[\mathsf{X}]$ *is a finitely generated* $\partial^*(\mathbb{C}[\partial\mathsf{X}])$-*module.*

Proof. As mentioned earlier, conditions (3) and (4) are equivalent by Proposition 5.1 and Corollary 5.2 of [1].

If $\dim(\mathsf{X}) > \dim(\partial\mathsf{X})$ then $\mathbb{C}[\mathsf{X}]$ is transcendental over $\partial^*(\mathbb{C}[\partial\mathsf{X}])$ and so condition (3) cannot hold. The inequality $\dim(\mathsf{X}) > \dim(\partial\mathsf{X})$ also implies that X contains a curve C to which Theorem 7 applies. This yields a closed essential surface detected by X. Hence, conditions (2) and (3) both imply that $\dim(\mathsf{X}) = \dim(\partial\mathsf{X})$.

We have established that each of the four conditions imply that $\dim(\mathsf{X}) = \dim(\partial\mathsf{X})$. By Proposition 2.4 of [6], $\dim(\mathsf{X}) \geq 1$. Also $\dim(\partial\mathsf{X}) \leq 1$, otherwise for any slope α on ∂N, there is a curve in X to which Theorem 7 can be applied to give an essential surface in N with boundary slope α. This contradicts Hatcher's theorem [15] that N has only finitely many boundary slopes. So, if any condition (1) through (4) holds, then $\dim(\mathsf{X}) = \dim(\partial\mathsf{X}) = 1$.

The theorem now follows from Theorems 6, 7, and Corollary 2.8. □

Given a trace function $I_\gamma \in \mathbb{C}[\mathsf{X}]$, let $\mathbb{C}[I_\gamma]$ denote the $\mathbb{C}$-subalgebra of $\mathbb{C}[\mathsf{X}]$ generated by I_γ. Consider the regular map $I_\gamma \colon \mathsf{X} \to \mathbb{C}$ and the induced map $I_\gamma^* \colon \mathbb{C}[x] \to \mathbb{C}[\mathsf{X}]$. If I_γ is non-constant on X then I_γ^* is injective and $\mathbb{C}[I_\gamma]$ is naturally isomorphic to $\mathbb{C}[x]$. Moreover, $\mathbb{Z}[x]$ is naturally isomorphic to the $\mathbb{Z}$-submodule $\mathbb{Z}[I_\gamma] \subseteq T(\mathsf{X})$ generated by all finite powers of I_γ. For instance, if α is a slope which is not detected by X, since $\dim(\mathsf{X}) \geq 1$, Theorem 7 implies that I_α is non-constant on X.

Theorem 9. *Suppose N is a knot manifold, X is an irreducible algebraic subset of $\mathsf{X}(N)$, and α is a slope. The following are equivalent.*

    (1) *X does not detect a closed essential surface and α is not detected by X.*
    (2) *$\mathbb{C}[\mathsf{X}]$ is integral over $\mathbb{C}[I_\alpha]$.*
    (3) *As a $\mathbb{C}[I_\alpha]$-module, $\mathbb{C}[\mathsf{X}]$ is a finitely generated and free.*

Proof. As with conditions (3) and (4) of the previous theorem, we know that (3) implies (2) and that (2) implies that $\mathbb{C}[\mathsf{X}]$ is a finitely generated $\mathbb{C}[I_\alpha]$-module. $\mathbb{C}[\mathsf{X}]$ is an integral domain so, as a $\mathbb{C}[I_\alpha]$-module, $\mathbb{C}[\mathsf{X}]$ is torsion free. $\mathbb{C}[I_\alpha]$ is a PID so $\mathbb{C}[X]$ is a free $\mathbb{C}[I_\alpha]$-module.

It remains to show that (1) and (2) are equivalent.

Assume first that (2) holds. Theorem 8 shows that X does not detect a closed essential surface and $\dim(\mathsf{X}) = 1$. Hence I_α is not constant. We have the regular map $I_\alpha \colon \mathsf{X} \to \mathbb{C}$ and $\overline{I_\alpha(\mathsf{X})} = \mathbb{C}$. Since $\mathbb{C}[\mathsf{X}]$ is integral over $\mathbb{C}[I_\alpha]$, Theorem 6 shows that $I_\alpha \colon \mathsf{X} \to \mathbb{C}$ does not have a hole. By Theorem 7, α is not strongly detected by X. Since X does not detect a closed essential surface, α cannot be weakly detected by X.

Now suppose that (1) holds. As in the proof of Theorem 8, we have that $\dim(\mathsf{X}) = 1$. Consider the regular map $I_\alpha \colon \mathsf{X} \to \mathbb{C}$. Either $\overline{I_\alpha(\mathsf{X})} = \mathbb{C}$ or I_α is constant on X. But since $\dim(\mathsf{X}) = 1$, Theorem 7 implies that if I_α is constant on X then α is strongly detected by X or X detects a closed essential surface. So we must have $\overline{I_\alpha(\mathsf{X})} = \mathbb{C}$.

By Theorem 6, it suffices to show that the regular map I_α does not have a hole. Again, we appeal to Theorem 7 and notice that if I_α has a hole then either α is strongly detected or X detects a closed essential surface. $\qquad\square$

Definition 3.6. Let $\mathsf{X} \subseteq \mathsf{X}(N)$ be an irreducible algebraic subset. The $\mathbb{C}$-*rank function* is the map

$$\mathrm{Rk}_{\mathsf{X}}^{\mathbb{C}} \colon \mathcal{S} \to \mathbb{Z}^+ \cup \{\infty\}$$

where $\mathrm{Rk}_{\mathsf{X}}^{\mathbb{C}}(\alpha)$ is the rank of $\mathbb{C}[\mathsf{X}]$ as a $\mathbb{C}[I_\alpha]$-module.

Remarks 2. *Using Theorem 9, we make the following observations.*

    (1) *If X does not detect a closed essential surface, $\mathrm{Rk}_{\mathsf{X}}^{\mathbb{C}}(\alpha) = \infty$ if and only if α is strongly detected by X.*
    (2) *X detects a closed essential surface if and only if $\mathrm{Tr}_{\mathsf{X}}^{\mathbb{C}}(\mathcal{S}) = \{\infty\}$.*
    (3) *By Lemma 1.4.4 of [9], there are only finitely many boundary slopes strongly detected by X. Avoiding this finite set, choose $\alpha \in \mathcal{S}$. Then X detects a closed essential surface if and only if $\mathrm{Rk}_{\mathsf{X}}^{\mathbb{C}}(\alpha) = \infty$.*

For X an irreducible algebraic component of $\mathsf{X}(N)$ we define X° to be the union of the irreducible components X' of $\mathsf{X}(N)$ such that

(1) X' detects a closed essential surface if and only if X detects a closed essential surface and

(2) for every slope α, X' detects α if and only X detects α.

Proposition 3.7. *If X is an irreducible algebraic component of $\mathsf{X}(N)$ then X° is defined over $\mathbb{Q}$.*

Proof. By a theorem of Weil [17, Ch. III, Thm. 7], it suffices to prove that X° is invariant under the action of $\mathrm{Aut}(\mathbb{C})$ on $\mathsf{X}(N)$ (see [6, Prop. 2.3] and [3, Section 5]).

First, we assume that X does not detect a closed essential surface. Theorem 9 implies that X° is the union of the irreducible algebraic components X' of $\mathsf{X}(N)$ such that, for every slope α, $\mathbb{C}[\mathsf{X}']$ is integral over $\mathbb{C}[I_\alpha]$ if and only if $\mathbb{C}[\mathsf{X}]$ is integral over $\mathbb{C}[I_\alpha]$.

Let $\phi \in \mathrm{Aut}(\mathbb{C})$, X' an irreducible component of X°, and $\alpha \in \mathcal{S}$. Define $\mathsf{X}'' = \phi(\mathsf{X}')$. The set $\phi(\mathsf{X}')$ is an irreducible algebraic component of $\mathsf{X}(N)$. To prove the proposition we need only show that $\mathbb{C}[\mathsf{X}'']$ is integral over $\mathbb{C}[I_\alpha]$ if and only if $\mathbb{C}[\mathsf{X}']$ is integral over $\mathbb{C}[I_\alpha]$.

The automorphism ϕ determines an automorphism on any complex polynomial ring by acting on the coefficients. This automorphism descends to an isomorphism $\phi\colon \mathbb{C}[\mathsf{X}'] \to \mathbb{C}[\mathsf{X}'']$. Since $\phi\colon \mathbb{C} \to \mathbb{C}$ restricts to the identity on $\mathbb{Q}$ and I_α is represented by a polynomial with integer coefficients we have $\phi(I_\alpha) = I_\alpha$. Therefore if $p \in \mathbb{C}[I_\alpha]$ then $\phi(p) \in \mathbb{C}[I_\alpha]$. So if $f \in \mathbb{C}[\mathsf{X}']$ is integral over $\mathbb{C}[I_\alpha]$, we can apply ϕ to an integral dependence relation for f over $\mathbb{C}[I_\alpha]$ to obtain one for $\phi(f)$ over $\mathbb{C}[I_\alpha]$. To prove the converse, we apply ϕ^{-1} to any integral dependence relation for $\mathbb{C}[\mathsf{X}'']$ over $\mathbb{C}[I_\alpha]$.

The case when X does detect a closed essential surface follows from the above argument along with Lemma 5.3 of [3]. $\qquad\square$

Definition 3.8. Suppose $\mathsf{X} \subseteq \mathsf{X}(N)$ is an algebraic set. The *rational trace ring* $T_\mathbb{Q}(\mathsf{X})$ is the smallest $\mathbb{Q}$-algebra that contains $T(\mathsf{X})$.

Theorem 10. *Suppose N is a knot manifold, X is an irreducible algebraic subset of $\mathsf{X}(N)$, and $\alpha \in \mathcal{S}$. The following are equivalent.*

(1) *X does not detect a closed essential surface and α is not detected by X.*

(2) *$T_\mathbb{Q}(\mathsf{X})$ is integral over $\mathbb{Q}[I_\alpha]$.*

(3) *As a $\mathbb{Q}[I_\alpha]$-module, $T_\mathbb{Q}(\mathsf{X})$ is a finitely generated and free.*

Proof. The first paragraph of the proof of Theorem 9 shows that (2) and (3) are equivalent. Theorem 9 shows that (2) implies (1).

It remains to show that (1) implies (2). Suppose then that X does not detect a closed essential surface and α is not detected by X. $T_\mathbb{Q}(\mathsf{X})$ is generated by $T(\mathsf{X})$ as a $\mathbb{Q}$-algebra, so it suffices to show that elements of $T(\mathsf{X})$ are integral over $\mathbb{Q}[I_\alpha]$.

Let $\{\mathsf{X}_j\}_1^n$ be the set of irreducible algebraic components of X° and take $f \in T(\mathsf{X})$. The map $\mathbb{C}[\mathsf{X}^\circ] \to \mathbb{C}[\mathsf{X}]$ induced by inclusion is given by restriction and is surjective. Take $F \in \mathbb{C}[\mathsf{X}^\circ]$ such that $F|_\mathsf{X} = f$. Let $\pi\colon \mathsf{X}^\circ \to \mathbb{C}^2$ be the map (F, I_α). Let (x, y) be the coordinates on $\mathrm{Im}(\pi)$ determined by $x\pi = F$ and $y\pi = I_\alpha$.

Fix $j \in \{1, \ldots, n\}$. By Theorem 8, $I_\alpha\colon \mathsf{X}_j \to \mathbb{C}$ is non-constant and $\overline{\pi(\mathsf{X}_j)}$ is an irreducible plane curve. Let $P_j \in \mathbb{C}[x, y]$, be an irreducible polynomial which defines $\overline{\pi(\mathsf{X}_j)}$. X° is defined over $\mathbb{Q}$ (Proposition 3.7) and π is given by polynomials

with $\mathbb{Z}$-coefficients, so we may assume that

$$P \;=\; \prod_{j=1}^{n} P_j \;\in \mathbb{Z}[x,y].$$

For each j, Theorem 9 gives that $F|_{\mathsf{X}_j}$ is integral over $\mathbb{C}[I_\alpha]$. This means that there is a polynomial in $\mathbb{C}[x,y]$ which is zero on $\overline{\pi(\mathsf{X}_j)}$ and monic as a polynomial in $(\mathbb{C}[x])[y]$. Select one such polynomial p_j with minimal degree. Then p_j is irreducible and must differ from P_j only by multiplication by some $\alpha_j \in \mathbb{C}$. We have

$$P \;=\; \prod P_j \;=\; \left(\prod \alpha_j\right) \cdot \left(\prod p_j\right).$$

Hence $\prod \alpha_j \in \mathbb{Z}$ and $(\prod \alpha_j^{-1}) \cdot P$ gives an integral dependence relation for f over $\mathbb{Q}[I_\alpha]$. $\qquad\square$

Definition 3.9. Suppose $\mathsf{X} \subseteq \mathsf{X}(N)$ is an irreducible algebraic subset. The $\mathbb{Q}$-*rank function* is the map

$$\mathrm{Rk}_{\mathsf{X}}^{\mathbb{Q}} \colon \; \mathcal{S} \to \mathbb{Z}^+ \cup \{\infty\}$$

where $\mathrm{Rk}_{\mathsf{X}}^{\mathbb{Q}}(\alpha)$ is the rank of $T_{\mathbb{Q}}(\mathsf{X})$ as a $\mathbb{Q}[I_\alpha]$-module.

It will require more work to show that there are related theorems which work over $\mathbb{Z}$. We begin by establishing some lemmas concerning valuations.

4. Valuations

Lemma 4.1. *Suppose $\nu \colon F^\times \to \Lambda$ is a valuation on the field F. Assume that*

$$cx^k \;=\; \sum_{i=0}^{k-1} a_i x^i$$

for some $c, x, a_i \in F$ and $k \in \mathbb{Z}^+$ such that $c \neq 0$, $\nu(c) = 0$, and $\nu(a_i) \geq 0$ for every i. Then $\nu(x) \geq 0$.

Proof. If every a_i is zero then $x = 0$ and so $\nu(x) = \infty$, so we may assume that at least one of the a_i's is non-zero. Let j be such that $\nu(a_j) + j\nu(x)$ is minimal in the set $\{\nu(a_i) + i\nu(x) \mid 0 \leq i < k\}$. We have

$$k\nu(x) \;=\; \nu(c) + k\nu(x) \;\geq\; \nu(a_j) + j\nu(x)$$

so

$$(k-j)\nu(x) \;\geq\; \nu(a_j) \;\geq\; 0.$$

Hence $\nu(x) \geq 0$. $\qquad\square$

Lemma 4.2. *If ν is a non-trivial valuation on $\mathbb{Q}$ then ν is a p-adic valuation.*

Proof. Let R be the valuation ring corresponding to ν and let $\mathfrak{m}$ be the maximal ideal in R. We know that $\mathbb{Z} \subset R$. Note that $\mathfrak{m} \cap \mathbb{Z}$ is prime (maximal) in $\mathbb{Z}$ so ν is p-adic. $\qquad\square$

The following lemma is essentially Lemma 2.2 of [7].

Lemma 4.3. *Suppose $p \in \mathbb{Z}$ is a prime and $h \in \mathbb{Z}[t]$ is irreducible over $\mathbb{Z}$ and of the form*

$$h(t) \;=\; c_0 + c_1 t + \cdots + c_{k-1} t^{k-1} + c_k p^r t^k$$

where $(c_k, p) = 1$ and $c_k \neq \pm 1$. Let $b \in \mathbb{C}$ be any root of h. Then there is a valuation ν on $\mathbb{Q}(b)$ with $\nu(p) = 0$ and $\nu(b) < 0$.

Proof. Since $(c_k, p) = 1$ and $h \in \mathbb{Z}[t]$ we know that r is an integer and at least zero. Now, for $i \in \{0, \dots, k-1\}$, define $a_i = c_i p^{k-1-i}$. We have $p^{(k-1)r} c_i = a_i(p^r)^i$, so

$$p^{(k-1)r} h(t) \ = \ a_0 + a_1(p^r t) + \dots + a_{k-1}(p^r t)^{k-1} + c_k(p^r t)^k$$

Set $d = p^r b$ and

$$h^*(t) \ = \ a_0 + a_1 t + \dots + a_{k-1} t^{k-1} + c_k t^k.$$

Then $h^* \in \mathbb{Z}[t]$ and $h^*(d) = 0$. We have

$$c_k d^k \ = \ \sum_{i=0}^{k-1} a_i d^i.$$

Let ν_p be an extension of the p-adic valuation to $\mathbb{Q}(b)$. Since $(c_k, p) = 1$, $\nu_p(c_k) = 0$. Also, for every i, $a_i \in \mathbb{Z}$ so $\nu(a_i) \geq 0$. Hence, Lemma 4.1 implies that $\nu_p(d) \geq 0$.

Since h is irreducible over $\mathbb{Z}$ and $\mathbb{Q}(b) = \mathbb{Q}(d)$, it follows that h^* is irreducible over $\mathbb{Z}$. Hence, neither b nor d is integral over $\mathbb{Z}$. By Corollary 5.2 of [1], we have a valuation ν on $\mathbb{Q}(b)$ with $\nu(d) < 0$. The restriction of ν to $\mathbb{Q}$ cannot be the p-adic valuation and (again using Lemma 4.1) cannot be trivial. By Lemma 4.2, $\nu|_{\mathbb{Q}} = \nu_q$ for some prime $q \in \mathbb{Z}$ which is different from p. Therefore, $\nu(p) = 0$. Also

$$\nu(b) \ = \ \nu(dp^{-r}) \ = \ \nu(d) - r\nu(p) \ = \ \nu(d) < 0.$$

$\square$

5. $T(\mathsf{X})$ AS A $\mathbb{Z}[I_\alpha]$-MODULE

Throughout this section, N is a knot manifold and Γ is its fundamental group. Next, we introduce some terminology from [20].

Definition 5.1. A representation $\rho \colon \Gamma \to \mathrm{SL}_2\mathbb{C}$ is *algebraic non-integral* if the image of ρ is in $\mathrm{SL}_2 F$, where F is a number field and $\chi_\rho(\gamma)$ is not an algebraic integer for some $\gamma \in \Gamma$. If ρ is an algebraic non-integral representation we abbreviate this by saying that ρ is an *ANI-representation*.

Similar to Theorem 7, the following result again follows from the work of Culler and Shalen. See Lemma 2 of [20].

Theorem 11. *For every ANI-representation ρ of Γ there is an associated non-empty essential surface $\Sigma \subset N$. Suppose that $\alpha \in \mathcal{S}$ is not a boundary slope. The surface Σ may chosen to have empty boundary if and only if $I_\alpha(\chi_\rho)$ is an algebraic integer.*

Definitions 5.2.
    (1) Suppose that ρ is an ANI-representation of Γ and $\Sigma \subset N$ is a non-empty essential surface. Σ is *associated to ρ* if it is contained in a surface given by ρ and Theorem 11.
    (2) Suppose that X is an algebraic subset of $\mathsf{X}(N)$. A surface Σ is *ANI-detected by X* if there is an ANI-representation ρ of Γ whose character lies on X and Σ is associated to ρ.

Proposition 5.3. *If X is an irreducible algebraic subset of $\mathsf{X}(N)$ and N contains a closed essential surface which is ANI-detected by X, then N contains a closed essential surface detected by X.*

Proof. Assume that X does not detect a closed essential surface and suppose for a contradiction that there is a closed essential surface in N associated to an ANI-representation $\rho\colon \Gamma \to \mathrm{SL}_2 F$ with $\chi_\rho \in \mathsf{X}$. Take $\gamma \in \Gamma$ with $\chi_\rho(\gamma)$ non-integral and choose $\alpha \in \mathcal{S}$ not a boundary slope.

By Theorems 3 and 11, there is a valuation ν on F with $\nu(I_\gamma(\chi_\rho)) < 0$ and $\nu(I_\alpha(\chi_\rho)) \geq 0$. By Theorem 10, there is a polynomial

$$f(x,y) \;=\; cx^r + \sum_{j=0}^{r-1} q_j(y)x^j$$

with $c \in \mathbb{Z} - \{0\}$, $q_j \in \mathbb{Z}[y]$, and $f(I_\gamma, I_\alpha) = 0$ on X. Since $\nu(I_\alpha(\chi_\rho)) \geq 0$, we know that $\nu(q_j(I_\alpha(\chi_\rho))) \geq 0$. But then, Lemma 4.1 implies that $\nu(I_\gamma(\chi_\rho)) \geq 0$, a contradiction. $\qquad\square$

Example 1. In [8] Cooper and Long ask whether there is a one-cusped manifold whose character variety detects a closed essential surface. This was answered by Tillmann in [25] where he presented a calculation which shows that the character variety of the Kinoshita-Terasaka knot (11n42) detects a closed essential surface. Together with data collected by Goodman, Heard, and Hodgson [12], http://www.ms.unimelb.edu.au/~snap/, Proposition 5.3 gives many more examples.

Suppose that N is a one-cusped hyperbolic 3-manifold and let X_0 be an algebraic component of $\mathsf{X}(N)$ which contains the character of a discrete faithful representation ρ_0. By Theorem 11 and Proposition 5.3, if ρ_0 is an ANI-representation then N contains a closed essential surface detected by $\mathsf{X}(N)$.

For all manifolds in the Callahan-Hildebrand-Weeks census of cusped hyperbolic manifolds with up to 7 tetrahedra [4], and for all complements of hyperbolic knots and links up to 12 crossings, Goodman, Heard, and Hodgson use the computer program `Snap` to attempt to (among other things) decide whether or not the holonomy representation is an ANI-representation. They find 252 such manifolds, 21 of which are complements of knots in S^3 (this is almost certainly not a complete list). The knots are 9a30, 9a31, 10a89, 10a96, 10a103, 11n97, 12n156, 12n245, 12n246, 12n260, 12n494, 12n508, 12n518, 12n600, 12n602, 12n604, 12n605, 12n694, 12n888, 12a1205, and 12a1288.

Roughly, the following theorem is proven by applying Theorem 10 and arguing as in [7].

Theorem 12. *Suppose N is a knot manifold, X is an irreducible algebraic subset of $\mathsf{X}(N)$, and $\alpha \in \mathcal{S}$. The following are equivalent.*

    (1) *X does not detect a closed essential surface and α is not detected by X.*
    (2) *$T(\mathsf{X})$ is integral over $\mathbb{Z}[I_\alpha]$.*
    (3) *$T(\mathsf{X})$ is a finitely generated $\mathbb{Z}[I_\alpha]$-module.*

Proof. As with Theorem 10, we need only argue that (1) implies (2). Suppose that (1) is true and take $\gamma \in \Gamma$. Our goal is to show that I_γ is integral over $\mathbb{Z}[I_\alpha]$, so we may assume that I_γ is not everywhere zero on X. By Theorem 8, $\dim(\mathsf{X}) = 1$ and by Theorem 9, I_α must be non-constant on X.

By Theorem 10, there is a polynomial $f(y,z) \in \mathbb{Z}[y,z]$ of the form

$$f(y,z) \;=\; cy^r + \sum_{j=0}^{r-1} q_j(z)y^j$$

with $q_j \in \mathbb{Z}[z]$ such that $c \neq 0$, $f(I_\gamma, I_\alpha)$ is the zero function on X, and the greatest common divisor of the coefficients of f is one. Choose f among all such polynomials to have smallest possible total degree. Then, since I_α and I_γ are not everywhere zero on X, neither y nor z divides f.

Take $k \in \mathbb{N}$ large enough such that $f(y, xy^{-k})$ has no two terms with the same degree in y. Take $l \in \mathbb{Z}$ as small as possible such that

$$g(x,y) \;=\; y^l f(y, xy^{-k})$$

is a polynomial in $\mathbb{Z}[x,y]$. Then $g(x,y)$ is of the form

$$g(x,y) \;=\; cy^s + \sum_{j=0}^{s-1} a_j x^{t_j} y^j$$

where the t_j's are distinct integers, the greatest common divisor of the a_j's is one, $g(I_\alpha I_\gamma^k, I_\gamma) = 0$ on X, and neither x nor y divides g.

We know that I_α is non-constant on X so there must be a point $\hat{x} \in \widetilde{\mathsf{X}}$ with $I_\alpha(\hat{x}) = 0$. If $I_\alpha I_\gamma^k$ is constant on X then $I_\alpha = K I_\gamma^{-k}$ for some $K \in \mathbb{C}$. Hence, $\hat{x}$ is an ideal point of X. By Theorem 2.2.1 and Proposition 2.3.1 of [10], either X detects a closed essential surface or an essential surface with boundary slope α. Both are contradictions so $I_\alpha I_\gamma^k$ must be non-constant on X. Therefore, we may choose a prime p such that $\gcd(c, p) = 1$ and for every root b of $g(p, y)$ there is a character $\chi_\rho \in \mathsf{X}$ with $\chi_\rho(\alpha) \cdot \chi_\rho(\gamma)^k = p$ and $\chi_\rho(\gamma) = b$.

Take

$$g(p,y) \;=\; \eta \cdot \prod_{j=0}^{u} h_j(y)$$

a complete factorization of $g(p, y)$ over $\mathbb{Z}$. So $\eta \in \mathbb{Z}$ and each h_j is irreducible over $\mathbb{Z}$. We claim that c divides η. The leading coefficient of $g(p, y)$ is c. If we denote the the leading coefficient of h_j as c_j, we have $c = \eta c_0 \cdots c_u$. If c doesn't divide η then there is some c_i with $\gcd(\alpha, c_i) > 1$. Write $c_i = p^t d$ where $\gcd(p, d) = 1$. Since $\gcd(c, p) = 1$ we must have $\gcd(c, d) > 1$, in particular $d \neq \pm 1$. Let b be a root of h_i. By our choice of p, there is $\chi_\rho \in \mathsf{X}$ which corresponds to the solution $g(p, b)$. Using Hilbert's Nullstellensatz, we may assume that the entries of the matrices in $\rho(\pi_1 N)$ lie in a number field F. By Lemma 4.3, we have a valuation ν on $\mathbb{Q}(b)$ with $\nu(p) = 0$ and $\nu(b) < 0$. Take an extension of ν to F. We have $\nu(I_\gamma(\chi_\rho)) < 0$ and

$$0 \;=\; \nu\left(I_\alpha(\chi_\rho) I_\gamma(\chi_\rho)^k\right) \;=\; \nu\left(I_\alpha(\chi_\rho)\right) + k \cdot \nu\left(I_\gamma(\chi_\rho)\right).$$

Since $k \geq 0$ we must have $\nu(I_\alpha(\chi_\rho)) \geq 0$. Hence, N contains a closed essential surface which is ANI-detected by X. Proposition 5.3 shows that this is a contradiction, so we must have that c divides η.

The constant η divides every coefficient of $g(p, y)$ and c divides η, so c divides every coefficient of $g(p, y)$. But the coefficient for z^j is $a_j p^{t_j}$ and $\gcd(c, p) = 1$. Hence c divides each a_j. The only integers that divide every a_j are ± 1. Therefore $f(I_\gamma, I_\alpha)$ is an integral dependence relation for I_γ over $\mathbb{Z}[I_\alpha]$. $\qquad\square$

Definition 5.4. Let $\mathsf{X} \subseteq \mathsf{X}(N)$ be an irreducible algebraic subset. The $\mathbb{Z}$-*rank function* is the map

$$\mathrm{Rk}_{\mathsf{X}}^{\mathbb{Z}} \colon \mathcal{S} \to \mathbb{Z}^+ \cup \{\infty\}$$

where $\mathrm{Rk}_{\mathsf{X}}^{\mathbb{Z}}(\alpha)$ is the rank of $T(\mathsf{X})$ as a $\mathbb{Z}[I_\alpha]$-module.

Corollary 5.5. *Suppose N is a knot manifold, X is an irreducible algebraic subset of $\mathsf{X}(N)$ which is defined over $\mathbb{Q}$. The following are equivalent.*

    (1) X *does not detect a closed essential surface and $\alpha \in \mathcal{S}$ is not detected by X.*
    (2) *Every $f \in T(\mathsf{X})$ satisfies a monic <u>irreducible</u> polynomial with coefficients in $\mathbb{Z}[I_\alpha]$.*

Proof. By Theorem 9, need only argue that (1) implies (2). Assume that X does not detect a closed essential surface and let f be a non-zero element of $T(\mathsf{X})$. By Theorem 8, X is a curve. Since X does not detect a closed essential surface, I_α is non-constant on X. Let V be the Zariski closure of the image of X under the map (f, I_α). V must be an irreducible affine curve defined over $\mathbb{Q}$. Let $p \in \mathbb{Q}[x, y]$ be a defining equation for V. By multiplying by an integer we may assume $p \in \mathbb{Z}[x, y]$. The proof of Theorem 12, shows that p is an integral dependence relation of f over $\mathbb{Z}[I_\alpha]$. Since V is irreducible, p is irreducible. $\qquad\square$

Definition 5.6. A knot manifold is called *small* if it does not contain a closed essential surface.

Corollary 5.7. *Let X be an irreducible algebraic subset of $\mathsf{X}(N)$, where N is a small knot manifold. A slope α is detected by X if and only if $\mathrm{Rk}_{\mathsf{X}}^{\mathbb{Z}}(\alpha) \neq \infty$.*

6. $\mathrm{PSL}_2\mathbb{C}$-CHARACTER VARIETIES

The theorems from Sections 3 and 5, also apply in the $\mathrm{PSL}_2\mathbb{C}$ setting after making appropriate definitions and using virtually identical arguments. See [2] for details in what follows.

As usual, let N be a compact irreducible orientable 3-manifold with torus boundary and let $\Gamma = \pi_1 N$. Denote the set of $\mathrm{PSL}_2\mathbb{C}$-representations of Γ as $\overline{\mathsf{R}}(N)$. $\mathrm{PSL}_2\mathbb{C}$ acts on $\overline{\mathsf{R}}(N)$ by conjugation. Let $\bar{t}\colon \overline{\mathsf{R}}(N) \to \mathsf{Y}(N)$ denote the corresponding algebro-geometric quotient. $\overline{\mathsf{R}}(N)$ and $\mathsf{Y}(N)$ are affine algebraic sets and the map $\bar{t}$ is a surjective regular map. $\mathsf{Y}(N)$ is called the $\mathrm{PSL}_2\mathbb{C}$-character variety for N and a point in $\mathsf{Y}(N)$ is called a $\mathrm{PSL}_2\mathbb{C}$-character. As with the $\mathrm{SL}_2\mathbb{C}$-character variety, every irreducible algebraic component of $\mathsf{Y}(N)$ has dimension at least one [6].

We would like to relate $\mathsf{Y}(N)$ to $\mathrm{SL}_2\mathbb{C}$-character varieties and define $T(\mathsf{Y})$ where Y is an irreducible algebraic subset of $\mathsf{Y}(N)$. First we claim that there is a well-defined equivalence class of $\mathbb{Z}_2$-central extensions

$$(2) \qquad\qquad 1 \to \mathbb{Z}_2 \to \hat{\Gamma} \to \Gamma \to 1$$

and a finite regular map from the $\mathrm{SL}_2\mathbb{C}$-character variety $\mathsf{X}(\hat{\Gamma})$ of $\hat{\Gamma}$ to $\mathsf{Y}(N)$ which contains Y in its image.

First note that if $\bar{\rho}_1, \bar{\rho}_2 \in \overline{\mathsf{R}}(N)$ have the same image under $\bar{t}$ then the representations are either both reducible or are both irreducible. Hence we may refer to the points of Y as being either reducible or irreducible.

Suppose that every element of Y is reducible and choose $\bar{\chi} \in \mathsf{Y}$ smooth in $\mathsf{Y}(N)$. There is a diagonal representation $\bar{\rho} \in \bar{t}^{-1}(\bar{\chi})$. This representation determines an extension (2) and a lift $\hat{\rho}\colon \hat{\Gamma} \to \mathrm{SL}_2\mathbb{C}$ of $\bar{\rho}$. Moreover, the isomorphism class of $\hat{\Gamma}$ is independent of our choices of $\bar{\chi}$ and $\bar{\rho}$. The natural epimorphisms $\mathrm{SL}_2\mathbb{C} \to \mathrm{PSL}_2\mathbb{C}$ and $\hat{\Gamma} \to \Gamma$ together induce a regular map $\phi\colon \mathsf{X}(\hat{\Gamma}) \to \mathsf{Y}(N)$. In fact, the image of ϕ is the quotient of $\mathsf{X}(\hat{\Gamma})$ under the natural action of $H^1(\hat{\Gamma}; \mathbb{Z}_2)$. Moreover, $\mathsf{Y} \subseteq \mathrm{Im}(\phi)$.

Otherwise Y contains an irreducible character which is a smooth point of $\mathsf{Y}(N)$. Let $\bar{\chi}$ be such a point and $\bar{\rho} \in \bar{t}^{-1}$. Exactly as before, we obtain an extension (2) and a lift $\hat{\rho}$ of $\bar{\rho}$ to $\hat{\Gamma}$. Again, the isomorphism class of $\hat{\Gamma}$ is independent of our choices, we obtain a finite regular map $\phi \colon \mathsf{X}(\hat{\Gamma}) \to \mathsf{Y}(N)$, the image of ϕ is the quotient of $\mathsf{X}(\hat{\Gamma})$ by $H^1(\hat{\Gamma}; \mathbb{Z}_2)$, and $\mathsf{Y} \subseteq \mathrm{Im}(\phi)$.

In either case, the induced map $\phi^* \colon \mathbb{C}[\mathrm{Im}(\phi)] \to \mathbb{C}[\mathsf{X}(\hat{\Gamma})]$ is injective. We define the trace ring $T(\mathrm{Im}(\phi))$ to be

$$T(\mathrm{Im}(\phi)) \;=\; (\phi^*)^{-1}\left(\mathrm{Im}(\phi^*) \cap T\big(\mathsf{X}(\hat{\Gamma})\big)\right).$$

The inclusion $\mathsf{Y} \to \mathrm{Im}(\phi)$ induces an epimorphism $\mathbb{C}[\mathrm{Im}(\phi)] \to \mathbb{C}[\mathsf{Y}]$. Define $T(\mathsf{Y})$ to be the image of $T(\mathrm{Im}(\phi))$ under this epimorphism.

Note that if $\gamma \in \Gamma$ and $\hat{\gamma}_1, \hat{\gamma}_2 \in \hat{\Gamma}$ are the preimages of γ then $I^2_{\hat{\gamma}_1} = I^2_{\hat{\gamma}_2}$. It follows that there is a well-defined squared trace function $I^2_\gamma \in T(\mathsf{Y})$.

To obtain results in the $\mathrm{PSL}_2\mathbb{C}$-setting corresponding to those in Sections 3 and 5, we use the $\mathrm{PSL}_2\mathbb{C}$-definitions rather than $\mathrm{SL}_2\mathbb{C}$-definitions and replace I_α with the squared trace function I^2_α whenever I_α is not in the $\mathrm{PSL}_2\mathbb{C}$-trace ring.

Remark 3. *The equivalence classes of extensions (2) are in bijective correspondence with the elements of $H^2(\Gamma; \mathbb{Z}_2)$.*

7. Ranks and examples

Throughout this section we assume that N is a knot manifold and $\Gamma = \pi_1 N$. For an irreducible algebraic set $\mathsf{X} \subseteq \mathsf{X}(N)$, observe that

(3) $$\mathrm{Rk}^{\mathbb{C}}_{\mathsf{X}} \;\leq\; \mathrm{Rk}^{\mathbb{Q}}_{\mathsf{X}} \;\leq\; \mathrm{Rk}^{\mathbb{Z}}_{\mathsf{X}}.$$

Among other things, this section gives examples which show that these inequalities need not be equalities.

If R is a ring, M is an R-module, and $S \subset M$ then we write $\langle S \rangle_R$ to denote the R-module generated by S.

7.1. **Abelian curves.** Suppose that $H_1(N; \mathbb{Z}) \cong \mathbb{Z}$. There is a curve $\mathsf{X}_A \subseteq \mathsf{X}(N)$ consisting of all characters of representations which factor through $H_1(N; \mathbb{Z})$. Let $\alpha \in \Gamma$ be a peripheral element whose image in $H_1(N; \mathbb{Z})$ generates $H_1(N; \mathbb{Z})$.

Every character in X_A is the character of a diagonal representation and the image subgroup in $\mathrm{SL}_2\mathbb{C}$ is generated by the image of α. It follows that every trace function $I_\alpha \in T(\mathsf{X}_A)$ is represented by a polynomial in $\mathbb{Z}[I_\alpha]$. Using the single coordinate I_α on X_A, we have $\mathsf{X}_A = \mathbb{C}$. In particular, we have the following well-known proposition.

Proposition 7.1. *Suppose that N is a knot manifold and $H_1(N; \mathbb{Z}) \cong \mathbb{Z}$. Let $\mathsf{X}_A \subseteq \mathsf{X}(N)$ be the curve consisting of all abelian representations of $\pi_1 N$. Then*

   (1) *X_A has exactly one ideal point. This ideal point detects the unique boundary slope which is trivial in $H_1(N; \mathbb{Z})$.*
   (2) *$T(\mathsf{X}_A) = \mathbb{Z}[I_\alpha]$.*

So, by the proposition, $T(\mathsf{X}_A)$ is a free $\mathbb{Z}[I_\alpha]$-module with rank one.

7.2. **Norm curves.** Recall the following definition from [3].

Definition 7.2. An algebraic curve component X of $\mathsf{X}(N)$ is called a *norm curve component* if $I_\gamma\colon \mathsf{X} \to \mathbb{C}$ is non-constant for every non-trivial peripheral element of Γ.

For example, if N is hyperbolic then there is a hyperbolic curve $\mathsf{X}_0 \subseteq \mathsf{X}(N)$. By Proposition 3.1.1 of [10], X_0 is a norm curve component.

Proposition 7.3. *Suppose that N is a knot manifold and X is an irreducible algebraic subset of $\mathsf{X}(N)$. Let α be a slope. If X is a norm curve component then $\mathrm{Rk}_{\mathsf{X}}^{\mathbb{C}}(\alpha) \geq 2$.*

Proof. Suppose that X is a norm component and that $\mathrm{Rk}_{X}^{\mathbb{C}}(\alpha) < 2$. By Theorem 8, $\dim(\mathsf{X}) = 1$. Hence, $\mathrm{Rk}_{\mathsf{X}}^{\mathbb{C}}(\alpha) = 1$ and we have $f \in \mathbb{C}[\mathsf{X}]$ with $f \cdot \mathbb{C}[I_\alpha] = \mathbb{C}[\mathsf{X}]$. In particular, f has an inverse $f^{-1} \in \mathbb{C}[I_\alpha]$. We have

$$\mathbb{C}[I_\alpha] \;=\; f^{-1}f \cdot \mathbb{C}[I_\alpha] \;=\; f^{-1} \cdot \mathbb{C}[\mathsf{X}] \;=\; \mathbb{C}[\mathsf{X}].$$

Therefore, X has exactly one ideal point. On the other hand, [3, Proposition 4.5] and Theorem 7 show that X strongly detects at least two boundary slopes, so X must have more than one ideal point. $\qquad\square$

7.3. **Two-bridge knots.** Suppose that K is a two bridge knot. Define $N = S^3 - K$ and $\Gamma = \pi_1 N$. Let $\langle \mu, \beta \,|\, \omega\beta = \mu\omega \rangle$ be the standard presentation for Γ as given in Section 4.5 of [18]. The element $\mu \in \Gamma$ is a meridian for the knot.

Suppose that X is an algebraic component of $\mathsf{X}(N)$ which is defined over $\mathbb{Q}$. Let $x = I_\mu$ and $y = I_{\mu\beta}$ in $T(\mathsf{X})$. It is shown in [14] that N is small and the slope represented by μ is not a boundary slope. By Corollary 5.5, we have a set of polynomials $\{p_j\}_{j=0}^{n}$ in $\mathbb{Z}[x]$ where X is given as the zero set in $\mathbb{C}^2$ of

$$P(x,y) \;=\; y^n - \sum_{j=0}^{n} p_j(x)y^j$$

and P is irreducible in $\mathbb{C}[x,y]$. Let $\mathcal{B} = \{y^j\}_{j=0}^{n-1} \subset T(\mathsf{X})$. Since P is zero in $T(\mathsf{X})$, $\mathcal{B}$ generates $T(\mathsf{X})$ as a $\mathbb{Z}[x]$-module, $T_\mathbb{Q}(\mathsf{X})$ as a $\mathbb{Q}[x]$-module, and $\mathbb{C}[\mathsf{X}]$ as a $\mathbb{C}[x]$-module. In particular,

$$(4) \hspace{4cm} \mathrm{Rk}_{\mathsf{X}}^{\mathbb{Z}}(\mu) \;\leq\; n.$$

Proposition 7.4. *$\mathcal{B}$ is a free basis for $\mathbb{C}[\mathsf{X}]$ as a $\mathbb{C}[x]$-module, $T_\mathbb{Q}(\mathsf{X})$ as a $\mathbb{Q}[x]$-module, and $T(\mathsf{X})$ as a $\mathbb{Z}[x]$-module.*

Proof. First, we induct on k to show that $M_k = \langle 1, y, \ldots, y^k \rangle_{\mathbb{C}[x]}$ is free of rank $k+1$ whenever $k \leq n-1$. The base case is immediate since $M_0 = \mathbb{C}[x]$.

Since $M_k = M_{k-1} + \langle y^k \rangle$, to see that $M_k = M_{k-1} \oplus \langle y^k \rangle$, it is enough to show that $M_{k-1} \cap \langle y^k \rangle = \{0\}$. Let f be an element of $M_{k-1} \cap \langle y^k \rangle$. Then we have $\{p_j\}_{j=0}^{k} \subset \mathbb{C}[x]$ such that

$$f \;=\; \sum_{j=0}^{k-1} p_j(x)y^j \;=\; p_k(x)y^k$$

Hence,

$$-p_k(x)y^k + \sum_{j=0}^{k-1} p_j(x)y^j \;\in\; (P) \subset \mathbb{Q}[x,y].$$

But $k < n$ and P is irreducible, so each p_j must be zero.

Therefore, $\mathcal{B}$ is a free basis for $\mathbb{C}[\mathsf{X}]$ as a $\mathbb{C}[x]$-module and using the inequalities (3) and (4) we have

$$\mathrm{Rk}_{\mathsf{X}}^{\mathbb{C}}(\mu) \;=\; \mathrm{Rk}_{\mathsf{X}}^{\mathbb{Q}}(\mu) \;=\; \mathrm{Rk}_{\mathsf{X}}^{\mathbb{Z}}(\mu) \;=\; n.$$

$\mathbb{Q}[x]$ is a PID, so $\mathcal{B}$ is a free basis for $T_{\mathbb{Q}}(\mathsf{X})$ as a $\mathbb{Q}[x]$-module.

To see that $\mathcal{B}$ is a free basis for $T(\mathsf{X})$ as a $\mathbb{Z}[x]$-module, it remains only to show that this module is free. To do this, we show that $\langle 1, y, \ldots, y^{k-1}\rangle_{\mathbb{Z}[x]} \cap \langle y^k\rangle_{\mathbb{Z}[x]} = \{0\}$, whenever $k \le n - 1$. If f is in this intersection, then it is in $M_{k-1} \cap \langle y^k\rangle_{\mathbb{C}[x]}$ which we have already seen is trivial. $\qquad\square$

Example 2. Let K be a trefoil knot. A calculation shows that $\mathsf{X}(N) = \mathsf{X}_A \cup \mathsf{X}_0$, where X_0 is an irreducible curve. The polynomial P which defines X_0 is $y - 1$. Therefore

$$\mathrm{Rk}_{\mathsf{X}_0}^{\mathbb{C}}(\mu) \;=\; \mathrm{Rk}_{\mathsf{X}_0}^{\mathbb{Q}}(\mu) \;=\; \mathrm{Rk}_{\mathsf{X}_0}^{\mathbb{Z}}(\mu) \;=\; 1.$$

Proposition 7.3 reaffirms the well-known fact that X_0 is not a norm curve.

Example 3. Let K be the figure-eight knot. A calculation shows that $\mathsf{X}(N) = \mathsf{X}_A \cup \mathsf{X}_0$, where X_0 is a hyperbolic curve. The polynomial P which defines X_0 is $y^2 + (-1 - x^2)y + (-1 + 2x^2)$. Therefore

$$\mathrm{Rk}_{\mathsf{X}_0}^{\mathbb{C}}(\mu) \;=\; \mathrm{Rk}_{\mathsf{X}_0}^{\mathbb{Q}}(\mu) \;=\; \mathrm{Rk}_{\mathsf{X}_0}^{\mathbb{Z}}(\mu) \;=\; 2.$$

$H^1(N; \mathbb{Z}_2) \cong \mathbb{Z}_2$ with generator σ. $H^1(N; \mathbb{Z}_2)$ acts on X_0 by $\sigma(x, y) = (-x, y)$. It follows that $T(\mathsf{Y}_0)$ can be identified with the subring (with 1) of $T(\mathsf{X}_0)$ generated by x^2 and y. Moreover, Y_0 can be identified with the zeros of $y^2 + (-1 - \xi)y + (-1 + 2\xi)$ where $\xi = x^2$. We have

$$\mathrm{Rk}_{\mathsf{Y}_0}^{\mathbb{C}}(\mu) \;=\; \mathrm{Rk}_{\mathsf{Y}_0}^{\mathbb{Q}}(\mu) \;=\; \mathrm{Rk}_{\mathsf{Y}_0}^{\mathbb{Z}}(\mu) \;=\; 2.$$

Let $\lambda = [\beta, \mu^{-1}][\mu, \beta^{-1}]$. Then λ commutes with μ and represents the boundary slope determined by a Seifert surface. Since $\sigma(\lambda)$ is the identity, $l = I_\lambda \in T(\mathsf{Y}_0)$. If we take (l, ξ, y) as coordinates on Y_0 then the ideal determined by Y_0 is generated by the following polynomials in $\mathbb{C}[l, \xi, y]$.

$$\begin{aligned}
p_1 &= (-1 - 4l) + (-11 + 4l)y + (16 - l)y^2 - 7y^3 + y^4 \\
p_2 &= 2 - l - 5\xi + \xi^2 \\
p_3 &= 1 - 2\xi + y + \xi y - y^2 \\
p_4 &= (2l) + \xi + (5 - l)y + 5y^2 + y^3.
\end{aligned}$$

Let $M = \langle 1, y, y^2, y^3\rangle_{\mathbb{C}[l]}$. The first polynomial describes the image of Y_0 under the map $\chi \mapsto (L(\chi), y(\chi))$. Hence, p_1 is irreducible and M is free of rank 4. The last polynomial shows that $\xi^i y^j \in M$ for every $i, j \in \mathbb{Z}^+$. It follows that

$$\mathrm{Rk}_{\mathsf{Y}_0}^{\mathbb{C}}(\lambda) \;=\; \mathrm{Rk}_{\mathsf{Y}_0}^{\mathbb{Q}}(\lambda) \;=\; \mathrm{Rk}_{\mathsf{Y}_0}^{\mathbb{Z}}(\lambda) \;=\; 4.$$

Let $s = I_{\mu^2\lambda}$. Then $s \in T(\mathsf{Y}_0)$. Similar calculations show that $\{1, y, y^2, y^3, \xi\}$ generates $\mathbb{C}[\mathsf{Y}_0]$ as a $\mathbb{C}[s]$-module and that

$$\mathrm{Rk}_{\mathsf{Y}_0}^{\mathbb{C}}(\mu^2\lambda) \;=\; \mathrm{Rk}_{\mathsf{Y}_0}^{\mathbb{Q}}(\mu^2\lambda) \;=\; \mathrm{Rk}_{\mathsf{Y}_0}^{\mathbb{Z}}(\mu^2\lambda) \;=\; 5.$$

7.4. Other examples.

Example 4. Let N be the `Snappea` census manifold M_{003} from [4]. N is a finite volume hyperbolic 3-manifold with a single cusp. $\langle \gamma, \eta \mid \gamma\eta\gamma^{-2}\eta\gamma\eta^3 \rangle$ is a presentation for $\Gamma = \pi_1 N$. The elements $\mu = (\eta^2\gamma\eta\gamma)^{-1}$ and $\lambda = (\gamma\eta\gamma)^{-1}\eta\gamma\eta$ together generate a peripheral subgroup. Define

$$m = I_\mu \qquad\qquad x = I_\gamma \qquad\qquad y = I_\eta \qquad\qquad z = I_{\gamma\eta}$$

Let $\mathsf{X}_0 \subset \mathsf{X}(N)$ be a hyperbolic curve and let Y_0 be the image of X_0 in the $\mathrm{PSL}_2\mathbb{C}$-character variety. We can take (m, x, y, z) as coordinates on X_0. The ideal in $\mathbb{C}[m, x, y, z]$ determined by X_0 is generated by the following three polynomials

$$p_1 = z^4 - mz^2 - z^2 + 1$$
$$p_2 = -z^2 + m + y$$
$$p_3 = -z^3 + mz + z + x.$$

The first polynomial is irreducible and gives that $\langle 1, z, z^2, z^3 \rangle_{\mathbb{C}[m]}$ is free of rank 4. The last two polynomials show that $x, y \in \langle 1, z, z^2, z^3 \rangle_{\mathbb{C}[m]}$. It follows that $\mathbb{C}[\mathsf{X}_0] = \langle 1, z, z^2, z^3 \rangle_{\mathbb{C}[m]}$ and

$$\mathrm{Rk}^{\mathbb{C}}_{\mathsf{X}_0}(\mu) \; = \; \mathrm{Rk}^{\mathbb{Q}}_{\mathsf{X}_0}(\mu) \; = \; \mathrm{Rk}^{\mathbb{Z}}_{\mathsf{X}_0}(\mu) \; = \; 4.$$

Let $\zeta = z^2$. It is straightforward to check that $m \in T(\mathsf{Y}_0)$ and that (m, ζ) can be taken as coordinates on Y_0. As such, Y_0 is the zero locus of the polynomial $\zeta^2 + (-m - 1)\zeta + 1$. It follows that

$$\mathrm{Rk}^{\mathbb{C}}_{\mathsf{Y}_0}(\mu) \; = \; \mathrm{Rk}^{\mathbb{Q}}_{\mathsf{Y}_0}(\mu) \; = \; \mathrm{Rk}^{\mathbb{Z}}_{\mathsf{Y}_0}(\mu) \; = \; 2.$$

We see that these ranks achieve the smallest possible value for the rank of a norm curve. In contrast to Example 3, $\mathrm{Rk}^{\mathbb{C}}_{\mathsf{Y}_0}(\mu) < \mathrm{Rk}^{\mathbb{C}}_{\mathsf{X}_0}(\mu)$.

Example 5. Let N be the exterior of the knot 8_{20}. The knot 8_{20} is hyperbolic. Let $\mathsf{X}_0 \subset \mathsf{X}(N)$ be a hyperbolic curve. As outlined in the introduction to this paper, Γ is generated by a pair of elements μ and γ where μ is a meridian. Define $x = I_\mu$, $y = I_\gamma$, and $z = I_{\gamma\mu}$. The functions x, y, z give an embedding of X_0 into $\mathbb{C}^3$.

Let $\mathcal{G} = \{g_j\}_1^5 \subset \mathbb{Z}[x, y, z]$ where

$$g_1 = 1 + 5y + 7y^2 + 2y^3 - 2y^4 - y^5 - 2x^2 - 6yx^2 - 3y^2x^2 + y^4x^2 + x^4 + yx^4$$
$$g_2 = -x - 3yx - y^2x - y^3x + x^3 + yx^3 + y^2z + y^3z$$
$$g_3 = -1 - 4y - 3y^2 + y^3 + y^4 + x^2 + 2yx^2 - y^3x^2 + y^2xz$$
$$g_4 = x + 3yx + 2y^2x - x^3 - yx^3 - z - 2yz - y^2z + x^2z$$
$$g_5 = 2 + 6y - 2y^3 - 3x^2 - yx^2 + 2y^2x^2 + xz - 3yxz + z^2.$$

$\mathcal{G}$ is a Groebner basis for the ideal $(\mathcal{G})$ with respect to the pure lexicographic order on monomials in $\mathbb{C}[x, y, z]$ determined by the relationship $y < x < z$. Furthermore, $(\mathcal{G})$ is the kernel of the natural map $\mathbb{C}[x, y, z] \to \mathbb{C}[\mathsf{X}_0]$.

We use the symbol $\equiv$ to indicate congruence modulo $(\mathcal{G})$ in $\mathbb{C}[x, y, z]$ and we write $LM(f)$ for the leading monomial of $f \in \mathbb{C}[x, y, z]$ with respect to our chosen monomial order. For $1 \leq j \leq 5$, let $m_j = LM(g_j)$. Then $(m_j)_1^5 = (yx^4, y^3z, y^2xz, x^2z, z^2)$ and $m_j < m_{j+1}$.

Oertel's work in [19] shows that N is a small knot manifold, so Corollary 5.5 implies that there are integral dependencies for y and z over $\mathbb{Z}[x]$ given by irreducible

polynomials. The polynomial g_1 is the polynomial for y and and a quick calculation produces the polynomial

$$z^5 - 2xz^4 + (-2 + 3x^2)z^3 + (12x - 9x^3 + x^5)z^2 + (-18x^2 + 10x^4 - x^6)z + (6x^3 - 2x^5) \in (\mathcal{G})$$

for z. These polynomials also show that the set $\mathcal{B} = \{y^i z^j \mid 0 \le i, j \le 4\}$ is a generating set for $\mathbb{C}[\mathsf{X}_0]$ as a $\mathbb{C}[x]$-module, $T_{\mathbb{Q}}(\mathsf{X}_0)$ as a $\mathbb{Q}[x]$-module, and $T(\mathsf{X}_0)$ as a $\mathbb{Z}[x]$-module. However, $\mathcal{B}$ is to large to be a basis for any of these modules.

Theorem 13. *The set $\mathcal{B}' = \{1, y, y^2, y^3, z, y^2 z\}$ is a free basis for $\mathbb{C}[\mathsf{X}_0]$ as a $\mathbb{C}[x]$-module, $T_{\mathbb{Q}}(\mathsf{X}_0)$ as a $\mathbb{Q}[x]$-module, and $T(\mathsf{X}_0)$ as a $\mathbb{Z}[x]$-module. In particular,*

$$\mathrm{Rk}_{\mathsf{X}_0}^{\mathbb{C}}(\mu) \;=\; \mathrm{Rk}_{\mathsf{X}_0}^{\mathbb{Q}}(\mu) \;=\; \mathrm{Rk}_{\mathsf{X}_0}^{\mathbb{Z}}(\mu) \;=\; 6.$$

The proof of Theorem 13 follows from the following lemmas.

Lemma 7.5. *If $0 \le k \le 4$ then the rank of $\langle 1, y, \ldots, y^k \rangle_{\mathbb{C}[x]}$ is $k + 1$.*

Proof. The proof uses the same argument as Proposition 7.4. $\qquad\square$

Lemma 7.6. *If $k \ge 1$ then there is a polynomial $P \in \mathbb{C}[x, y]$ such that*

$$x^{2k} z \;\equiv\; P + z(1 + 2y + y^2)^k.$$

Proof. Induct on k and use the polynomial g_4. $\qquad\square$

Lemma 7.7. *If $k \ge 3$ then there is a polynomial $P \in \mathbb{C}[x, y]$ such that*

$$y^k z \;\equiv\; P \pm y^2 z.$$

Proof. Induct on k and use the polynomial g_2. $\qquad\square$

Lemma 7.8. *If $k \ge 1$ then there is a polynomial $P \in \mathbb{C}[x, y]$ and $d_0, \ldots, d_4 \in \mathbb{C}$ such that*

$$x^k z \;\equiv\; P + d_0 z + d_1 yz + d_2 y^2 z + d_3 xz + d_4 xyz.$$

Proof. The lemma is trivial if $k = 1$.

If k is even, a straightforward application of Lemmas 7.6 and 7.7 gives numbers d_0, d_1, and d_2 such that $x^k z$ is equivalent to a polynomial in

$$d_0 z + d_1 yz + d_2 y^2 z + \mathbb{C}[x, y].$$

If $k \ge 3$ is odd, we apply the result in the even case to obtain $c_0, c_1, c_2 \in \mathbb{C}$ where $x^k z$ is equivalent to a polynomial in

$$c_0 xz + c_1 xyz + c_2 xy^2 z + \mathbb{C}[x, y].$$

To finish the proof, notice that g_3 shows that $xy^2 z$ is represented by a polynomial in $\mathbb{C}[x, y]$. $\qquad\square$

Lemma 7.9. *Suppose $p \in \mathbb{C}[x, y]$, $g \in \mathbb{C}[x]$, and $d_0, \ldots, d_4 \in \mathbb{C}$. Define*

$$f \;=\; p + z \left(g + d_0 + d_1 y + d_2 y^2 + d_3 x + d_4 xy \right).$$

If $f \in (\mathcal{G})$ then $f = p$.

Proof. The proof is by induction on $\deg(g)$.

If $\deg(g) = 0$ then $g \in \mathbb{C}$ and

$$f \;=\; p + (g + d_0)z + d_1 yz + d_2 y^2 z + d_3 xz + d_4 xyz \;\in\; (\mathcal{G}).$$

Since $\mathcal{G}$ is a Groebner basis for $(\mathcal{G})$, some m_j must divide the leading term of the polynomial on the right. This forces $f = p$.

Let $k \geq 1$ and assume that the result holds for polynomials with degree less than k. Suppose that $g = \sum_0^k c_j x^j$. By Lemma 7.8, we have $q \in \mathbb{C}[x, y]$ and $r_0, \ldots, r_4 \in \mathbb{C}$ such that

$$c_k x^k z \; \equiv \; q + r_0 z + r_1 yz + r_2 y^2 z + r_3 xz + r_4 xyz.$$

Therefore,

$$p + q + \left(\sum_{j=0}^{k-1} c_j x^j \right) z + (d_0 + r_0)z + (d_1 + r_1)yz +$$
$$+ (d_2 + r_2)y^2 z + (d_3 + r_3)xz + (d_4 + r_4)xyz \; \in \; (\mathcal{G}).$$

By the inductive hypothesis, we have

$$c_2 = \cdots = c_{k-1} = 0$$
$$c_0 + d_0 + r_0 = 0$$
$$c_1 + d_3 + r_3 = 0$$
$$d_1 + r_1 = d_2 + r_2 = d_4 + r_4 = 0.$$

Hence,

$$f \; = \; p - r_0 z + d_1 yz + d_2 y^2 z - r_3 xz + d_4 xyz.$$

As before, we must have $f = p$ since $\mathcal{G}$ is a Groebner basis for $(\mathcal{G})$. $\qquad\square$

Lemma 7.10. *For every $k \in \mathbb{Z}^+$,*

$$\langle 1, y, \ldots, y^k, z \rangle_{\mathbb{C}[x]} \; = \; \langle 1, \ldots, y^k \rangle_{\mathbb{C}[x]} \oplus \langle z \rangle_{\mathbb{C}[x]}.$$

Proof. Take $h \in \langle 1, y, \ldots, y^k \rangle_{\mathbb{C}[x]} \cap \langle z \rangle_{\mathbb{C}[x]}$. Then there are polynomials $p \in \mathbb{C}[x, y]$ and $g \in \mathbb{C}[x]$ such that p represents h in $\mathbb{C}[\mathsf{X}_0]$ and $p \equiv -gz$. Define $f = p + gz$ and notice that $f \in (\mathcal{G})$. By Lemma 7.9, $p \in (\mathcal{G})$ and $h = 0$. $\qquad\square$

Lemma 7.11. *$\langle \mathcal{B}' \rangle_{\mathbb{C}[x]}$ is a free $\mathbb{C}[x]$-module with rank 6.*

Proof. Lemmas 7.5 and 7.10, reduce the problem to showing that the intersection

$$(5) \qquad\qquad \langle 1, \ldots, y^3, z \rangle_{\mathbb{C}[x]} \cap \langle y^2 z \rangle_{\mathbb{C}[x]}$$

is trivial.

Since $g_3 \in \mathcal{G}$, we have that $\langle 1, \ldots, y^3, xy^2 z \rangle_{\mathbb{C}[x]} = \langle 1, \ldots, y^4 \rangle_{\mathbb{C}[x]}$. Together with Lemmas 7.5 and 7.10, this shows that $\langle 1, \ldots, y^3, xy^2 z, z \rangle_{\mathbb{C}[x]}$ is free with rank six.

Now suppose that h is an element of the intersection (5). Then xh is an element of both $\langle 1, \ldots, y^3, z \rangle_{\mathbb{C}[x]}$ and $\langle xy^2 z \rangle_{\mathbb{C}[x]}$. But we know that the intersection of this module is trivial. Hence, $hx = 0$ and so $h = 0$. $\qquad\square$

Now to prove Theorem 13, we need only show that $\mathcal{B}'$ spans the modules in question. This is straightforward using the polynomials $\mathcal{G}$.

Example 6. N is the once punctured hyperbolic torus bundle from Section 5 of [11]. For more details in the calculations that follow, see [11]. We have

$$\Gamma = \left\langle \alpha, \beta, \tau \mid \tau\alpha\tau^{-1} = (\beta\alpha\beta)^{-1}, \tau\beta\tau^{-1} = \beta\alpha(\beta\alpha\beta)^{-3} \right\rangle.$$

The elements τ and $\lambda = [\alpha, \beta]$ form a basis for a peripheral subgroup of N. The functions

$$
\begin{array}{llll}
t = I_\tau & u = I_{\alpha\tau} & v = I_{\beta\tau} & w = I_{\alpha\beta\tau} \\
x = I_\alpha & y = I_\beta & z = I_{\alpha\beta} & l = I_\lambda
\end{array}
$$

give an embedding of $\mathsf{X}(N)$ into $\mathbb{C}^8$. $\mathsf{X}(N)$ contains exactly two hyperbolic curves, we label them X_ϵ where $\epsilon \in \{0, 1\}$. It is straightforward to verify that X_ϵ strongly detects the slopes represented by t and lt and does not detect a closed essential surface. This is done in [11]. It is also easy to use Theorem 9 to confirm these facts.

Theorem 14. *Let $\epsilon \in \{0, 1\}$.*

- (1) $\mathrm{Rk}^{\mathbb{C}}_{\mathsf{X}_\epsilon}(\lambda) = 8$
- (2) $\mathrm{Rk}^{\mathbb{Q}}_{\mathsf{X}_\epsilon}(\lambda) = 16$, *and*
- (3) $\mathrm{Rk}^{\mathbb{Z}}_{\mathsf{X}_\epsilon}(\lambda) = 17$.
- (4) *As a $\mathbb{Z}[l]$-module, $T(\mathsf{X}_\epsilon)$ is torsion free but not free.*

We prove Theorem 14 as a series of lemmas.

Suppose k is a field, let $R = k[x_0, \ldots, x_n]$, and $\mathcal{I} \subseteq R$ and ideal. Take $<$ to be the monomial order generated by $x_i < x_j$ iff $i < j$ and let $\mathcal{G} = \{g_j\}_1^r$ be a Groebner basis for $\mathcal{I}$ with respect to $<$. Write $m_j = LT(g_j)$. For $f \in R$, write $\bar{f}$ to denote its image in $R/\mathcal{I}$.

Lemma 7.12. *Suppose that $m_j \in k[x_1, \ldots x_n]$ for every j. If $\{n_j\}_1^k$ is a collection of monomials in $k[x_1, \ldots, x_n]$ such that*

- (1) *The $k[x_0]$-submodule of $R/\mathcal{I}$ generated by $\{\bar{n}_j\}_1^{k-1}$ is free of rank $k - 1$,*
- (2) *$n_i < n_j$ for every $i < j$, and*
- (3) *n_k is not divisible by any m_j*

then the $k[x_0]$-submodule generated by $\{\bar{n}_j\}_1^k$ is free of rank k.

Proof. Suppose $f \in R$ and

$$
\bar{f} \in \langle \bar{n}_1, \ldots, \bar{n}_{k-1} \rangle_{k[x_0]} \cap \langle \bar{n}_k \rangle_{k[x_0]}.
$$

Then there is a set of polynomials $\{p_j\}_1^k \subset k[x_0]$ such that

$$
\bar{f} = -p_k \bar{n}_k = \sum_{j=1}^{k-1} p_j \bar{n}_j.
$$

In particular

$$
g := \sum_{j=1}^{k} p_j n_j \in \mathcal{I}.
$$

If $p_k = 0$ then $\bar{f} = 0$ and we are done. Otherwise, let cx_0^s be its leading term. Then, by (2), $LT(g) = cx_0^s n_k$. Since $g \in \mathcal{I}$ and $\mathcal{G}$ is a Groebner basis for $\mathcal{I}$, we must have that $LT(g)$ is divisible by some m_j. This contradicts (3), so we must have $p_k = 0$. $\square$

Let $\mathcal{I}$ be the kernel of the natural map $\mathbb{C}[l, t, u, v, w, x, y, z] \to \mathbb{C}[\mathsf{X}_\epsilon]$. The following polynomials are in $\mathcal{I}$

$$8u + tyz - ty^3z \qquad\qquad 4v - 6ty + ty^3$$
$$4w + 2tz - ty^2z \qquad\qquad 2x - yz.$$

Remark 4. *Recall that Groebner bases can be used to solve the ideal membership problem.*

This shows that the projection given by $(l, t, u, v, w, x, y, z) \mapsto (l, t, y, z)$, restricts to an isomorphism (defined over $\mathbb{Q}$) from $\mathsf{X}_0 \cup \mathsf{X}_1$ onto its image. Hence, we have natural identifications of $\mathbb{C}[\mathsf{X}_\epsilon]$ and $\mathbb{C}[\mathsf{X}_0 \cup \mathsf{X}_1]$ with quotients of $\mathbb{C}[l, t, y, z]$. Define

$$g_1 = 16 + (-4 - 2l)z^2 + z^4$$
$$g_2 = (-8 - 2l) + z^2 + 2y^2$$
$$g_3 = 2t + (-1)^\epsilon \cdot iz^2$$
$$h = -8 + (2 + l)z^2 + 2t^2$$

and $\mathcal{G} = \{g_1, g_2, g_3\}$. Let $<$ be the pure lexicographic monomial order for $\mathbb{C}[l, t, y, z]$ induced by the relationship $l < z < y < t$. It can be easily verified that $\mathcal{G}$ is a Groebner basis for $(\mathcal{G})$ and $(\mathcal{G})$ is the kernel of the natural map $\mathbb{C}[l, t, y, z] \to \mathbb{C}[\mathsf{X}_\epsilon]$. Similarly, $\{g_1, g_2, h\}$ is a Groebner basis for the ideal it generates and this ideal is the kernel of the natural map $\mathbb{C}[l, t, y, z] \to \mathbb{C}[\mathsf{X}_0 \cup \mathsf{X}_1]$.

Lemma 7.13. *The set*

$$\mathcal{B}_\mathbb{C} \;=\; \{1, z, z^2, z^3, y, zy, z^2y, z^3y\}$$

is a basis for $\mathbb{C}[\mathsf{X}_\epsilon]$ as a $\mathbb{C}[l]$-module.

Proof. Since the natural map $\mathbb{C}[l, t, y, z]/(\mathcal{G}) \to \mathbb{C}[\mathsf{X}_\epsilon]$ is an isomorphism, we need only show that $\mathcal{B}_\mathbb{C}$ is a $\mathbb{C}[l]$-module basis for $\mathbb{C}[l, t, y, z]/(\mathcal{G})$.

Let $\mathcal{M} \subseteq \mathbb{C}[l, t, y, z]/(\mathcal{G})$ be the $\mathbb{C}[l]$-submodule generated by $\mathcal{B}_\mathbb{C}$. Lemma 7.12 shows that $\mathcal{M}$ is free with rank 8.

The polynomial g_3 shows that t may be expressed in terms of z. So, to finish the proof, it is enough to show that $y^jz^k \in \mathcal{M}$ for every pair of integers $j, k \geq 0$. By definition of $\mathcal{B}_\mathbb{C}$ and g_1, $z^j \in \mathcal{M}$, for every integer $j \geq 0$. Using g_2, we also have that $y^2z^j \in \mathcal{M}$, for every integer $j \geq 0$. Now, using this fact and the definition of $\mathcal{B}_\mathbb{C}$, we conclude that if $f \in \mathcal{M}$ then $yf \in \mathcal{M}$. In particular, $y^jz^k \in \mathcal{M}$ for every pair of integers $j, k \geq 0$. Therefore, $\mathcal{M} = \mathbb{C}[l, y, z]/(\mathcal{G})$. $\square$

Lemma 7.14. *The set*

$$\mathcal{B}_\mathbb{Q} \;=\; \mathcal{B}_\mathbb{C} \cup t\mathcal{B}_\mathbb{C}$$

is a basis for $T_\mathbb{Q}(\mathsf{X}_\epsilon)$ as a $\mathbb{Q}[l]$-module.

Proof. The natural map $\mathbb{C}[l, t, y, z] \to \mathbb{C}[\mathsf{X}_\epsilon]$ restricts to a surjection $\mathbb{Q}[l, t, y, z] \to T_\mathbb{Q}(\mathsf{X}_\epsilon)$. The kernel of this map is $(\mathcal{G}) \cap \mathbb{Q}[l, t, y, z]$. We claim that $\{g_1, g_2, h\}$ is a Groebner basis for this ideal with respect to the monomial order $<$. It is immediate to verify that this is a Groebner basis for (g_1, g_2, h), so we need only show that $(g_1, g_2, h) = (g_1, g_2, g_3) \cap \mathbb{Q}[l, t, y, z]$. The inclusion $(g_1, g_2, h) \subseteq (g_1, g_2, g_3) \cap \mathbb{Q}[l, t, y, z]$ is clear. To prove the opposite, suppose $f \in (\mathcal{G}) \cap \mathbb{Q}[l, t, y, z]$. Then $f(\chi) = 0$ for every $\chi \in \mathsf{X}_\epsilon$. Observe that complex conjugation induces a bijection

$\mathsf{X}_0 \to \mathsf{X}_1$. So $f(\bar{\chi})$ is also zero. In particular, f is zero on $\mathsf{X}_0 \cup \mathsf{X}_1$. Therefore $f \in (g_1, g_2, h)$.

Let $\mathcal{M} \subset T_{\mathbb{Q}}(\mathsf{X}_\epsilon)$ be the $\mathbb{Q}[l]$-submodule generated by $\mathcal{B}_{\mathbb{Q}}$. Since $\{g_1, g_2, h\}$ is a Groebner basis for the kernel of the map $\mathbb{Q}[l, t, y, z] \to T_{\mathbb{Q}}(\mathsf{X}_\epsilon)$, Lemma 7.12 shows that $\mathcal{M}$ is free with rank 16. The remainder of the proof follows an argument similar to the last part of the proof of Lemma 7.13. $\qquad\square$

Let
$$\mathcal{B}_{\mathbb{Z}} = \{1, y, y^2, y^3, t, u, v, w, x, z, ty, uy, xy, xy^2, tx, ux, vy\}.$$

Lemma 7.15. $\mathcal{B}_{\mathbb{Z}} - \{vy\}$ *is a basis for* $T_{\mathbb{Q}}(\mathsf{X}_\epsilon)$ *as a* $\mathbb{Q}[l]$-*module.*

Proof. The following equations hold in $\mathbb{C}[\mathsf{X}_\epsilon]$.

$$z^2 = 2((4+l) - y^2) \qquad\qquad z^3 = 2((4+l)z - 2xy)$$
$$yz = 2x \qquad\qquad yz^2 = 2((4+l)y - y^3)$$
$$yz^3 = 2((8+2l)x - 2xy^2) \qquad\qquad tz = (l+4)w - 2uy$$
$$tz^2 = 2((l+4)t - 2ux) \qquad\qquad tz^3 = 2((l^2 + 6l + 4)w - 2(l+2)uy)$$
$$tyz = 2tx \qquad\qquad tyz^2 = 2((l-2)ty + 4v)$$
$$tyz^3 = 4((l+2)tx - 4u)$$

So, every element of $\mathcal{B}_{\mathbb{Q}}$ can be expressed as a $\mathbb{Q}[l]$-linear combination of the elements from $\mathcal{B}_{\mathbb{Z}} - \{vy\}$. Since the rank of this module equals $|\mathcal{B}_{\mathbb{Z}} - \{vy\}|$, this set is a basis. $\qquad\square$

Lemma 7.16. $\mathcal{B}_{\mathbb{Z}}$ *generates* $T(\mathsf{X}_\epsilon)$ *as a* $\mathbb{Z}[l]$-*module.*

Proof. The following equations hold in $\mathbb{C}[\mathsf{X}_\epsilon]$. It is tedious but straightforward to use them to show that $\mathcal{B}_{\mathbb{Z}}$ generates $T(\mathsf{X}_\epsilon)$ as a $\mathbb{Z}[l]$-module.

$$y^2 z = 2yx \qquad x^2 u = 2u + xt \qquad uy^2 x = (6+l)xu - (6+l)t$$
$$yz = 2x \qquad x^2 = -y^2 + (6+l) \qquad yxt = (6+l)w - 2uy$$
$$yw = 2u \qquad y^4 = (6+l)(y^2 - 2) \qquad zy^2 x = 2(-y^3 + (6+l)y)$$
$$zyu = 2xu \qquad zxu = (6+l)w - 2uy \qquad zyx = 2(-y^2 + (6+l))$$
$$y^3 z = 2y^2 x \qquad zx = (4+l)y - y^3 \qquad vyu = 3y^2 x - 2(6+l)x$$
$$zu = yt \qquad u^2 = (-6-l) + y^2 \qquad vy^2 x = (6+l)(3w - yu)$$
$$xw = yt \qquad tyu = 2xy - (6+l)z \qquad wy^3 = 2((4+l)u - xt)$$

$$t^2 = (-4 - 6l - l^2) + (2+l)y^2 \qquad t^2 x = (-4 - 6l - l^2)x + (2+l)y^2 x$$
$$t^2 y = (-4 - 6l - l^2)y + (2+l)y^3 \qquad txu = (-12 - 8l - l^2) + (4+l)y^2$$
$$uxt = (-12 - 8l - l^2) + (4+l)y^2 \qquad vxt = 2(l+3)yx - (l+6)(l+1)z$$
$$wxt = (-4 - 6l - l^2)y + (2+l)y^3 \qquad vyt = 2((l+3)y^2 - (l+1)(l+6))$$
$$vxu = (-6 - 7l - l^2)y + (3+l)y^3 \qquad tv = (l+1)y^3 - (5l + l^2)y$$
$$v^2 = (-12l - 2l^2) + (3+2l)y^2$$

$$zw = 2t \qquad\qquad y^2u = (4+l)u - xt \qquad\qquad x^2y = -y^3 + (6+l)y$$

$$wy^2 = 2uy \qquad\qquad y^3z = (6+l)(yx-z) \qquad\qquad x^2t = -2xu + (6+l)t$$

$$wyx = 2ux \qquad\qquad xv = (6+l)w - 3uy \qquad\qquad uy^3 = (6+l)(yu-w)$$

$$xyu = 3ty - 2v \qquad\qquad uv = 3yx - (6+l)z \qquad\qquad uw = -(4+l)y + y^3$$

$$x^2y^2 = 2(6+l) \qquad\qquad tw = 2yx - (4+l)z \qquad\qquad wyu = 2(y^2 - (6+l))$$

$$ty^2x = 4u + 2xt \qquad\qquad wxu = 2yx - (6+l)z \qquad\qquad wyt = 2(y^2x - (4+l)x)$$

$$wy^2x = 6yt - 4v \qquad\qquad u^2y = (-6-l)y + y^3 \qquad\qquad u^2x = (-6-l)x + y^2x$$

$$vyx = -lu + 3xt \qquad\qquad 2vy = (6+l)t - lxu \qquad\qquad tu = y^2x - (4+l)x$$

$$yxu = 3ty - 2v \qquad\qquad vy^2 = (3-l)yt + lv \qquad\qquad w^2 = 2((-4-l) + y^2)$$

$$zv = 2xt - 2u \qquad\qquad zxt = (l-2)yt + 4v \qquad\qquad wv = 2y^2x - 2(3+l)x \quad \square$$

Lemma 7.17. *Define*

$$\mathcal{M}_1 \;=\; \Big\langle \mathcal{B}_{\mathbb{Z}} - \{t, ux, vy\} \Big\rangle_{\mathbb{Z}[l]} \subset T(\mathsf{X}_\epsilon) \quad and \quad \mathcal{M}_2 \;=\; \Big\langle t, ux, vy \Big\rangle_{\mathbb{Z}[l]} \subset T(\mathsf{X}_\epsilon).$$

Then $T(\mathsf{X}_\epsilon) = \mathcal{M}_1 \oplus \mathcal{M}_2$.

Proof. Lemma 7.16 shows that $T(\mathsf{X}_\epsilon) = \mathcal{M}_1 + \mathcal{M}_2$, so we need only show that $\mathcal{M}_1 \cap \mathcal{M}_2 = \{0\}$.

Suppose $f \in \mathcal{M}_1 \cap \mathcal{M}_2$ and label the elements of $\mathcal{B}_{\mathbb{Z}} - \{t, ux, vy\}$ as $\{b_j\}_1^{14}$. Then we have $\{p_j\}_1^{14} \cup \{q_j\}_1^3 \subset \mathbb{Z}[l]$ with

$$f \;=\; \sum_{j=1}^{14} p_j b_j \;=\; q_1 t + q_2 ux + q_3 vy.$$

This yields the expression

$$2q_3 vy \;=\; -2q_1 t - 2q_2 ux + 2\sum_{j=1}^{14} p_j b_j.$$

However, $\mathcal{B}_{\mathbb{Z}} - \{vy\}$ is a $\mathbb{Q}[l]$-basis for $T_{\mathbb{Q}}(\mathsf{X}_\epsilon)$, so the only way to write $2vy$ as a $\mathbb{Z}[l]$-linear combination in $\mathcal{B}_{\mathbb{Z}} - \{vy\}$ is listed in the proof of Lemma 7.16, namely

$$2vy \;=\; (6+l)t - lux.$$

Hence $p_j = 0$ for every $1 \le j \le 14$. $\qquad\qquad\qquad\qquad\qquad\qquad \square$

Lemma 7.18. $\mathcal{M}_2$ *is not projective.*

Proof. Let

$$\pi \colon \bigoplus_1^3 \mathbb{Z}[l] \to \mathcal{M}_2$$

be the epimorphism given by $(p, q, r) \mapsto pt + qux + rvy$. Argue as in the proof of Lemma 7.17 to see that $\ker(\pi)$ is generated by $(l+6, -l, -2)$. Let $\mathcal{J}$ be the ideal $(2, l) \subset \mathbb{Z}[l]$ and view $\mathcal{J}$ as a $\mathbb{Z}[l]$-module. The map

$$\eta \colon \bigoplus_1^3 \mathbb{Z}[l] \to \mathcal{J}$$

given by $(p, q, r) \mapsto -2q + lr$ is also an epimorphism. We have $\ker(\pi) \subseteq \ker(\eta)$, so there is an epimorphism
$$\gamma \colon \mathcal{M}_2 \to \mathcal{J}$$
with $\gamma\pi = \eta$. In particular,
$$\gamma(t) = 0 \qquad\qquad \gamma(ux) = -2 \qquad\qquad \gamma(vy) = l.$$
Let
$$\phi \colon \mathbb{Z}[l] \oplus \mathbb{Z}[l] \to \mathcal{J}$$
be the epimorphism given by $(p, q) \mapsto 2p + lq$.

Suppose for a contradiction, that $\mathcal{M}_2$ is projective. Then γ lifts to an epimorphism
$$\psi \colon \mathcal{M}_2 \to \mathbb{Z}[l] \oplus \mathbb{Z}[l].$$
For $1 \le j \le 3$, we have $(p_j, q_j) \in \mathbb{Z}[l] \oplus \mathbb{Z}[l]$ with
$$\psi(t) = (p_1, q_1) \qquad \psi(ux) = (p_2, q_2) \qquad \psi(vy) = (p_3, q_3).$$
Then
$$0 = \gamma(t) = \phi(p_1, q_1) = 2p_1 + lq_1$$
$$-2 = \gamma(ux) = \phi(p_2, q_2) = 2p_2 + lq_2$$
$$l = \gamma(vy) = \phi(p_3, q_3) = 2p_3 + lq_3.$$
Notice that the constant terms of p_1 and p_3 must be zero and the constant term of p_2 is -1. Apply ψ to the equation $2vy + lux = (6 + l)t$ to see that the equation
$$2p_3 + lp_2 = (6 + l)p_1$$
must hold in $\mathbb{Z}[l]$. For a contradiction, observe that the the degree one coefficient of the left hand side of this equation is odd and on the right hand side the coefficient is even. $\qquad \square$

Lemma 7.19. *The $\mathbb{Z}[l]$-module $T(\mathsf{X}_\epsilon)$ is torsion free but not free and* $\mathrm{Rk}^{\mathbb{Z}}_{\mathsf{X}_\epsilon}(\lambda) = 17$.

Proof. $\mathbb{C}[\mathsf{X}_\epsilon]$ is an integral domain, so $T(\mathsf{X}_\epsilon)$ is torsion free as a $\mathbb{Z}[l]$-module. Direct summands of free modules are projective. Therefore, by Lemmas 7.17 and 7.18, $T(\mathsf{X}_\epsilon)$ is not a free $\mathbb{Z}[l]$-module.

By Lemma 7.16, $\mathrm{Rk}^{\mathbb{Z}}_{\mathsf{X}_\epsilon}(\lambda) \le 17$. Suppose that $\{a_j\}_1^{16}$ generates $T(\mathsf{X}_\epsilon)$ as a $\mathbb{Z}[l]$-module. Then every element of $\mathcal{B}_\mathbb{Z}$ can be written as a $\mathbb{Z}[l]$-linear combination of the a_j's. By Lemma 7.14, $\{a_j\}_1^{16}$ is a free $\mathbb{Q}[l]$-basis for $T_\mathbb{Q}(\mathsf{X}_\epsilon)$. It follows that the $\mathbb{Z}[l]$-module generated by $\{a_j\}_1^{16}$ is also free. This is a contradiction because we know that $T(\mathsf{X}_\epsilon)$ is not a free $\mathbb{Z}[l]$-module. Therefore, $\mathrm{Rk}^{\mathbb{Z}}_{\mathsf{X}_\epsilon}(\lambda) = 17$. $\qquad \square$

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